Chapter 1.2 Propositional Logic

Reading Assignment: 1.2
Next Class: 1.3

Motivation

- Propositions: statements.
- Propositional wffs:
- Propositional logic/Statement logic/Propositional calculus: A formal system from formal logic to see how to reach logic conclusions based on given statements.
- Propositional logic provides a formalism to translate sentences into formal symbols.
  - Can analyze sentences without the “clutter” introduced by the “meaning” and ambiguity of natural language.
  - Truth values are absolute. There is no “maybe” or “probably”.
  - A set of well defined operators allows to express interrelations among statements without ambiguity.
Motivation (cont’d)

• Example:
  If the birds are flying south and the leaves are turning, then it must be fall. Fall brings cold weather. The leaves are turning but the weather is not cold. Therefore the birds are not flying south.

Propositions/Statements:
  B: Birds are flying south
  L: The leaves are turning
  F: It must be fall.
  C: The weather is cold

Propositional wffs:
  \((B \land L) \rightarrow F\)
  \(F \rightarrow C\)
  \(L \land C'\)

Conclusion:
  B’

Propositional Logic

• Deriving a logical conclusion by combining many propositions and using formal logic: hence, determining the truth of arguments.

• Definition of Argument:
  An argument is a sequence of statements in which the conjunction of the initial statements (called the premises/hypotheses) is said to imply the final statement (called the conclusion).

• An argument can be presented symbolically as
  \((P_1 \land P_2 \land ... \land P_n) \rightarrow Q\)
  where \(P_1, P_2, ..., P_n\) represent the hypotheses and \(Q\) represents the conclusion.
  -- Note: \(P\) and \(Q\) are general, compound statements and can thus take forms such as \((A \rightarrow B) \rightarrow C\) or \(A \lor B'\)
Valid Argument

• What is a valid argument? $\Leftrightarrow$ When does Q logically follow from $P_1, P_2, ..., P_n$.

• Informal answer: Whenever the truth of hypotheses leads to the conclusion

• Note: We need to focus on the relationship of the conclusion to the hypotheses and not just any knowledge we might have about the conclusion Q.

• Example:
  – $P_1$: Neil Armstrong was the first human to step on the moon.
  – $P_2$: Mars is a red planet
    And the conclusion
  – $Q$: No human has ever been to Mars.
  – This wff $P_1 \Lambda P_2 \rightarrow Q$ is not a tautology

Valid Argument

• **Definition of valid argument:**

  An argument is valid if and only if it is a tautology.

  An argument is valid if whenever the hypotheses are all true, the conclusion must also be true. A valid argument is intrinsically true, i.e. $(P_1 \Lambda P_2 \Lambda ... \Lambda P_n) \rightarrow Q$ is a tautology.

  – Arguments are valid solely based on their structure. The “meaning” of the original sentences is irrelevant.
Valid Argument

• The validity of an argument can be tested in different ways:
  – Tautology test using truth tables
    + Simple and straightforward to do.
    - Table becomes very big for larger arguments.
    - There are situations in formal logic where you can not write down truth tables. (We will see this later)
  – Proof sequences using propositional calculus

• Definition of **Proof Sequence**:
  A proof sequence is a sequence of wffs in which each wff is either a hypothesis or the result of applying one of the formal system’s derivation rules to earlier wffs in the sequence.

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Rules for Propositional Logic

• **Derivation rules for propositional logic**

<table>
<thead>
<tr>
<th>Equivalence Rules</th>
<th>Inference Rules</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allows individual wffs to be rewritten</td>
<td>Allows new wffs to be derived</td>
</tr>
<tr>
<td>Truth preserving rules</td>
<td>Work only in one direction</td>
</tr>
</tbody>
</table>

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Equivalence Rules

• These rules state that certain pairs of wffs are equivalent, hence one can be substituted for the other with no change to truth values.
• The set of equivalence rules are summarized here:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Equivalent to</th>
<th>Abbreviation for rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>R V S</td>
<td>S V R</td>
<td>Commutative - comm</td>
</tr>
<tr>
<td>(R V S) V Q</td>
<td>(R Λ S) Λ Q</td>
<td>Associative - ass</td>
</tr>
<tr>
<td>(R V S)’'</td>
<td>R’ Λ S’</td>
<td>De-Morgan’s Laws De-Morgan</td>
</tr>
<tr>
<td>R → S</td>
<td>R’ V S</td>
<td>implication - imp</td>
</tr>
<tr>
<td>R</td>
<td>(R’)’</td>
<td>Double Negation - dn</td>
</tr>
<tr>
<td>P&lt;-&gt;Q</td>
<td>(P → Q) Λ (Q → P)</td>
<td>Equivalence - equ</td>
</tr>
</tbody>
</table>

Inference Rules

• Inference rules allow us to add a wff to the last part of the proof sequence, if one or more wffs that match the first part already exist in the proof sequence.

<table>
<thead>
<tr>
<th>From</th>
<th>Can Derive</th>
<th>Abbreviation for rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>R, R → S</td>
<td>S</td>
<td>Modus Ponens- mp</td>
</tr>
<tr>
<td>R → S, S’</td>
<td>R’</td>
<td>Modus Tollens- mt</td>
</tr>
<tr>
<td>R, S</td>
<td>R Λ S</td>
<td>Conjunction- con</td>
</tr>
<tr>
<td>R Λ S</td>
<td>R, S</td>
<td>Simplification- sim</td>
</tr>
<tr>
<td>R</td>
<td>R V S</td>
<td>Addition- add</td>
</tr>
</tbody>
</table>

• Note: Inference rules do not work in both directions, unlike equivalence rules.
Examples

- *Example* for using Equivalence rule in a proof sequence:
  - Simplify \((A' \lor B') \lor C\)
    1. \((A' \lor B') \lor C\)
    2. \((A \land B)' \lor C\) 1, De Morgan
    3. \((A \land B) \rightarrow C\) 2, imp

- *Example* of using Inference Rule
  - If it is bright and sunny today, then I will wear my sunglasses. \((A \rightarrow B)\)

  **Modus Ponens**
  It is bright and sunny today. Therefore, I will wear my sunglasses.
  (From \(A, A \rightarrow B\), we have \(B\).)

  **Modus Tollens**
  I will not wear my sunglasses. Therefore, it is not bright and sunny today.
  (From \(B', A \rightarrow B\), we have \(A'\).)

Deduction Method

- To prove an argument of the form
  \[ P_1 \land P_2 \land \ldots \land P_n \rightarrow R \rightarrow Q \]

  **deduction method** allows for the use of \(R\) as an additional hypothesis
  and thus prove
  \[ P_1 \land P_2 \land \ldots \land P_n \land R \rightarrow Q \]

- Prove \((A \rightarrow B) \land (B \rightarrow C) \rightarrow (A \rightarrow C)\)

  Using deduction method, prove \((A \rightarrow B) \land (B \rightarrow C) \land A \rightarrow C\)
  1. \(A \rightarrow B\) hyp
  2. \(B \rightarrow C\) hyp
  3. \(A\) hyp
  4. \(B\) 1,3 mp
  5. \(C\) 2,4 mp

- The above is called the rule of **Hypothetical Syllogism** or hs in short.
- Many such other rules can be derived from existing rules which thus provide easier and faster proofs.
More Inference Rules

• These rules can be derived using the previous rules. They provide a faster way of proving arguments.

<table>
<thead>
<tr>
<th>From</th>
<th>Can Derive</th>
<th>Name / Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P \to Q, Q \to R )</td>
<td>( P \to R )</td>
<td>Hypothetical syllogism - hs</td>
</tr>
<tr>
<td>( P \lor Q, P' )</td>
<td>( Q )</td>
<td>Disjunctive syllogism - ds</td>
</tr>
<tr>
<td>( P \to Q )</td>
<td>( Q' \to P' )</td>
<td>Contraposition - cont</td>
</tr>
<tr>
<td>( Q' \to P' )</td>
<td>( P \to Q )</td>
<td>Contraposition - cont</td>
</tr>
<tr>
<td>( P )</td>
<td>( P \land P )</td>
<td>Self-reference - self</td>
</tr>
<tr>
<td>( P \lor P )</td>
<td>( P )</td>
<td>Self-reference - self</td>
</tr>
<tr>
<td>( (P \land Q) \to R )</td>
<td>( P \to (Q \to R) )</td>
<td>Exportation - exp</td>
</tr>
<tr>
<td>( P, P' )</td>
<td>( Q )</td>
<td>Inconsistency - inc</td>
</tr>
<tr>
<td>( P \land (Q \lor R) )</td>
<td>( (P \land Q) \lor (P \land R) )</td>
<td>Distributive - dist</td>
</tr>
<tr>
<td>( P \lor (Q \land R) )</td>
<td>( (P \lor Q) \land (P \lor R) )</td>
<td>Distributive - dist</td>
</tr>
</tbody>
</table>

Equivalence rule

Equivalence rule

Proofs of inference rules

• Prove that \((P \to Q) \to (Q' \to P')\) is a valid argument (called Contraposition – con).
  – Hence prove, \((P \to Q) \land Q' \to P'\) (using deduction method).
  – The above is true using the modus tollens inference rule.

• Prove \(P \land P' \to Q\) (called Inconsistency)
  – Proof 1:
    1. \(P\)          hyp
    2. \(P'\)          hyp
    3. \(P \lor Q\)    1, add
    4. \(Q \lor P\)    3, comm
    5. \((Q')' \lor P\) 4, dn
    6. \(Q' \to P\)    5, imp
    7. \((Q')'\)      2, 6, mt
    8. \(Q\)          7, dn
-- Proof 2:
1. $P$  
   hyp
2. $P'$  
   hyp
3. $P \lor Q$  
   1, add
4. $Q$  
   2, 3, ds

-- Proof 3:
1. Use truth table

<table>
<thead>
<tr>
<th>$P$</th>
<th>$P'$</th>
<th>$P \land P'$</th>
<th>$P \land P' \rightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Tautology

Proofs using Propositional Logic

• Prove the argument

$A \land (B \rightarrow C) \land [(A \land B) \rightarrow (D \lor C')] \land B \rightarrow D$

– First, write down all the hypotheses.
1. $A$
2. $B \rightarrow C$
3. $(A \land B) \rightarrow (D \lor C')$
4. $B$

Use the inference and equivalence rules to get at the conclusion $D$.
5. $C$  
   2,4, mp
6. $A \land B$  
   1,4, con
7. $D \lor C'$  
   3,6, mp
8. $C' \lor D$  
   7, comm
9. $C \rightarrow D$  
   8, imp
   and finally
10. $D$  
    5,9, mp

The idea is to keep focused on the result and sometimes it is very easy to go down a longer path than necessary.
More Proofs

• \((A \land B)' \land (C' \land A)' \land (C \land B')' \rightarrow A'\) is an argument

1. \((A \land B)'\) hyp
2. \((C' \land A)'\) hyp
3. \((C \land B')'\) hyp
4. \(A' \lor B'\) 1, De Morgan
5. \(B' \lor A'\) 4, comm
6. \(B \rightarrow A'\) 5, imp
7. \((C')' \lor A'\) 2, De Morgan
8. \(C' \rightarrow A'\) 7, imp
9. \(C' \lor (B')'\) 3, De Morgan
10. \((B')' \lor C'\) 9, comm
11. \(B' \rightarrow C'\) 10, imp
12. \(B' \rightarrow A'\) 8, 11, hs
13. \((B \rightarrow A') \land (B' \rightarrow A')\) 6, 12, con

Proof Continued

• At this point, we have now to prove that

\((B \rightarrow A') \land (B' \rightarrow A') \rightarrow A'\)

• Proof continues,

14. \(A \rightarrow B'\) 6, cont
15. \(A \rightarrow A'\) 14, 12, hs
16. \(A' \lor A'\) 15, imp
17. \(A'\) 16, self
Proving Verbal Arguments

• Russia was a superior power, and either France was not strong or Napoleon made an error. Napoleon did not make an error, but if the army did not fail, then France was strong. Hence the army failed and Russia was a superior power.
• Converting it to a propositional form using letters A, B, C and D
  A: Russia was a superior power
  B: France was strong        B’: France was not strong
  C: Napoleon made an error   C’: Napoleon did not make an error
  D: The army failed          D’: The army did not fail
• Combining, the statements using logic
  \((A \land (B’ \lor C))\) hypothesis
  C’ hypothesis
  \((D’ \rightarrow B)\) hypothesis
  \((D \land A)\) conclusion
• Combining them, the propositional form is
  \((A \land (B’ \lor C)) \land C’ \land (D’ \rightarrow B) \rightarrow (D \land A)\)

Verbal Argument Proof

• Prove \((A \land (B’ \lor C)) \land C’ \land (D’ \rightarrow B) \rightarrow (D \land A)\)
• Proof sequence
  1. \(A \land (B’ \lor C)\) hyp
  2. C’ hyp
  3. \(D’ \rightarrow B\) hyp
  4. A 1, sim
  5. B’ \lor C 1, sim
  6. C \lor B’ 5, comm
  7. B’ 2, 6, ds
  8. \(B’ \rightarrow (D’)^{\prime}\) 3, cont
  9. \((D’)^{\prime}\) 7, 8, mp
  10. D 9, dn
  11. \(D \land A\) 4, 10, con
Class Exercise

• Prove the following arguments
  – \((A' \rightarrow B') \land (A \rightarrow C) \rightarrow (B \rightarrow C)\)
  – \((Y \rightarrow Z') \land (X' \rightarrow Y) \land (Y \rightarrow (X \rightarrow W)) \land (Y \rightarrow Z) \rightarrow (Y \rightarrow W)\)

• If the program is efficient, it executes quickly. Either the program is efficient, or it has a bug. However, the program does not execute quickly. Therefore it has a bug. (use letters E, Q, B)

• The crop is good, but there is not enough water. If there is a lot of rain or not a lot of sun, then there is enough water. Therefore the crop is good and there is a lot of sun. (use letters C, W, R, S)