Outline

• Subset Sum – Decimal

• Subset Sum – Binary

• Subset Sum – Unary
Subset Sum (Decimal)

Decision Version:

Input:

- $S$ - set of decimal integers
- $t$ - an integer

Output:

Yes, if there is a subset $S' \subseteq S$ s.t. $\sum x = t$, where $x \in S$

No, otherwise
Subset Sum (Decimal)

Subset Sum is NP Complete:

Step 1 - Subset sum is in NP
  • Certificate is the subset $S'$
  • Verification can be done in polynomial time.

Step 2 - Reduction
  • Reduction using 3-CNF
Subset Sum (Decimal)

Reduction:

3-CNF Formula $\rightarrow$ Subset Sum $\rightarrow$ 3-SAT

$S, t$ $\rightarrow$ Y, N
Reduction (contd):

• Input is 3-CNF formula with \( n \) variables and \( m \) clauses

• Eg : \( f = c1 \land c2 \land c3 \land c4 \)
  - \( C1 = (x1 \lor \neg x2 \lor \neg x3) \)
  - \( C2 = (\neg x1 \lor \neg x2 \lor \neg x3) \)
  - \( C3 = (\neg x1 \lor \neg x2 \lor x3) \)
  - \( C4 = (x1 \lor x2 \lor x3) \)

• Here \( m = 4 \) and \( n = 3 \)
• One satisfying assignment is \((x1 = 0, x2 = 0, x3 = 1)\)
Reduction (contd):

Table Construction

• For each variable $x$, create two numbers (rows) $x$ and $x'$
• For each clause $C_i$, create two numbers (rows) $C_i$ and $C_i'$
• All numbers are in base 10
  - This prevents carries during addition.
• Each number has $n+m$ digits
• Each digit corresponds to a clause or a variable.
• Leading $n$ digits are labeled by variables
  - A digit $x_i$ has 1 for variables $x_i$ and $x_i'$
• Trailing $m$ digits are labeled by clauses
  - A digit $C_i$ has 1 for variable $x_i$, if $x_i$ occurs in $C_i$
<table>
<thead>
<tr>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>C1</th>
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<tr>
<th>t</th>
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<th>1</th>
<th>4</th>
<th>4</th>
<th>4</th>
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</table>
Notes:

• **t** has 1 in each digit labeled by a variable and 4 in each digit labeled by a clause.
• All the numbers (rows) generated are unique.
• Greatest digit in any position is 6. (So any base \( b \geq 7 \) will work for this reduction)
• The table has 2(m+n) rows.
• The rows for clauses \((C_i \text{ and } C_i')\) act as make-up rows.

(to make the sum as 4) –

  ▪ For any clause \( C_i \) with 3 variables then in a satisfying assignment,
    ○ If one of those variables is true, then both rows \( C_i \) and \( C_i' \) make up and sum becomes 4
    ○ If two variables is true, then \( C_i \) makes up and sum becomes 4
    ○ If all three variables is true, then \( C_i' \) makes up and sum becomes 4
Notes:

• Transformation from 3-CNF to table is in polynomial time.
• 3 – CNF is satisfiable if and only if there is a subset $S'$ of $S$ whose sum is $t$.
• Only one of $x_i$ or $x_i'$ rows can be included in the subset.
Subset Sum (Binary)

Decision Version:

Input:
- \( S \) - set of binary integers
- \( t \) - an integer

Output:
Yes, if there is a subset \( S' \subseteq S \) s.t. \( \sum x = t \), where \( x \in S \)
No, otherwise

Notes:
- Use Subset sum (decimal) for reduction.
- Decimal to binary can be converted in polynomial time.
- If a decimal number has \( k \) digits, then its binary number has at most \( 3k \) digits.
Subset Sum (Binary)

Reduction:

Subset sum (decimal)

Subset Sum (binary)

S, t

S', t

Y

N
Subset Sum (Unary)

Unary representation:

\[ 1 = 1, \quad 11 = 2, \quad 111 = 3 \text{ and } 1111 = 4 \]

Decision Version:

Same as subset sum (binary) except integers are in unary representation.

- Conversion of an integer (in decimal/binary) to unary takes exponential time.
- Since conversion is not in constant time, we cannot use reduction technique to prove if it is NP-complete.

- Subset sum (Unary) is in P.
  - It can be solved using dynamic programming.
- Subset sum is a weakly NP complete problem as it is in P when encoded in unary and NP-Complete else (base -2,10)