Tighter bound for Coupon Collector using Union Bound

Since the bounds that we obtain from Markov and Chebyshev for Coupon Collector are not ”tight”, we will now use the Union bound to get better results.

Let E_i^r be the event that we have not yet obtained coupon i even after r trials. Then,

$$\begin{aligned} Pr[E_i^r] &= \left(1 - \frac{1}{n}\right)^r \\ &\leq e^{\frac{-r}{n}} \end{aligned} \tag{1}$$

We used the identity $\left(1 + \frac{t}{n}\right)^n \leq e^t$ above.

If we substitute $r = \beta n \ln n$,

$$\begin{aligned} Pr[E_i^{\beta n \ln n}] &\leq e^{\frac{-\beta n \ln n}{n}} \\ &= e^{-\beta \ln n} \\ &= \left(e^{\ln n}\right)^{-\beta} \\ &= n^{-\beta} \end{aligned} \tag{2}$$

We used the fact that $x^{ab} = (x^a)^b$ in the derivation to simplify the expression.

We want to find(bound) the probability of $P[X > r]$.

$$Pr[X > r] = Pr[\cup_{i=1}^n E_i^r] \tag{3}$$

$$\begin{aligned} &\leq \sum_{i=1}^n Pr[E_i^r] \\ &\leq \sum_{i=1}^n n^{-\beta} \\ &= n \times n^{-\beta} \end{aligned}$$

$$Pr[X > \beta n \ln n] = n^{-\beta+1} \tag{4}$$

We used union bound to approximate (3). ie The probability of union of E_i^r is less than or equal to the sum of the probabilities. We can see that bound obtained using union bound (4) is much tighter than the bounds that we could have derived from Markov and Chebyshev.