Lecture 3: Growth of Functions

Growth of Functions

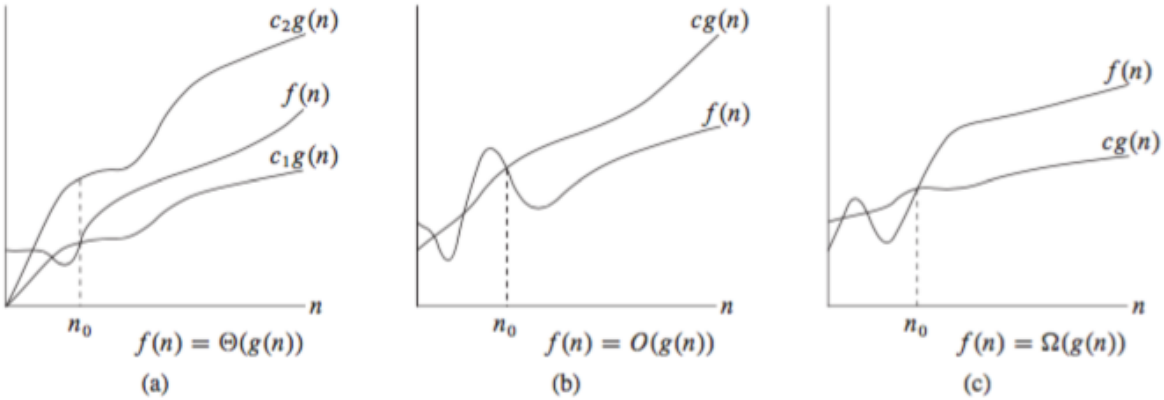
- Order of growth of the running time of an algorithm
  - Characterizes an algorithm's efficiency
  - Allows comparing relative performance of algorithms
- With large enough input sizes, in terms of n
  - Merge sort $\theta(n \lg n)$ beats insertion sort $\theta(n^2)$
- We study large enough input sizes to make relevant the order of growth of the running time
  - We study the asymptotic efficiency of algorithms
  - How running time of an algorithm increases with the size of the input in the limit, as input size increases without bound
- An asymptotically more efficient algorithm is usually the best choice for all but very small inputs

Asymptotic Notation

- Notation to describe asymptotic running time of an algorithm
  - Defined as functions with domain as $\mathbb{N} = \{0, 1, 2, \dots\}$
  - Describe worst case running time of $T(n)$
  - Usually defined only for integer input sizes
  - Careful not to abuse asymptotic notation
- Remember we characterize running time of algorithms with functions
  - Insertion sort: $\Theta(n^2)$
    - After abstraction from $an^2 + bn + c$
- $\Theta(n^2)$ stands for function $an^2 + bn + c$ as the worst case running time of insertion sort
- Asymptotic notation can also characterize other aspects of algorithms, i.e. amount of space

Θ-notation

- What does $T(n) = \Theta(n^2)$ means as worst-case running time for insertion sort?
- For a given function $g(n)$, we denote by $\Theta(g(n))$ the set of functions
- $\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and $n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \ \forall \ n \geq n_0\}$.
- A function $f(n)$ belongs to the set $\Theta(g(n))$ if there exist positive constants c_1 and c_2 such that it can be ?sandwiched? between $c_1 g(n)$ and $c_2 g(n)$, for sufficiently large n.
- $\Theta(g(n))$ is a set, then, $f(n) \in \Theta(g(n))$, but we write: $f(n) = \Theta(g(n))$
- For all $n \geq n_0$, $f(n)$ is equal to $g(n)$ to within a constant factor
- $g(n)$ is an asymptotically tight bound for $f(n)$
- By definition, $\Theta(g(n))$ requires every member $f(n) \in \Theta(g(n))$ be asymptotically nonnegative
  - $f(n)$ nonnegative when n is sufficiently large
- Using the formal definition
- Example using the Θ-notation for $\frac{1}{2}n^2 - 3n = \Theta(n^2)$
- Determine positive constants c_1, c_2, and n_0 such that $c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2$
  - dividing by n^2, $c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2$
  - For case $c_1 \leq 1/2 - 3/n$
  - Because c_1 is a positive constant, then, $0 < 1/2 - 3/n$, then $n > 6$
  - then, if $n_0 = 7$ then $c_1 \leq 1/2 - 3/7$, which is equal to $c_1 \leq 1/14$. Let $c_1 = 1/14$
- Determine positive constants c_1, c_2, and n_0 such that $c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2$
- For case $1/2 - 3/n \leq c_2$, when $n \to \infty$ then, $1/2 - 3/n \to 1/2$, then, $c_2 = 1/2$
  - For $c_1 = 1/14$, $c_2 = 1/2$ and $n_0 = 7$, it holds that $f(n) \in \Theta(n^2)$ or $\frac{1}{2}n^2 - 3n = \Theta(n^2)$



O-notation

- The O notation denotes an asymptotic upper bound
- For a function $g(n)$ we denote $O(g(n))$ and say: big-oh of g of n, or oh of g of n
- $O(g(n))$ also refers to a set of functions
- $O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n) \ \forall \ n \geq n_0\}$.
- Use O-notation to give upper bound on a function, to within a constant factor
- Note that $f(n) = \Theta(g(n))$ implies $f(n) = O(g(n))$
  - Θ-notation is stronger than O-notation

Ω-notation

- Ω-notation provides an asymptotic lower bound
- For a function $g(n)$, we denote $\Omega(g(n))$ and say: big-omega of g of n or omega of g of n
- $\Omega(g(n))$ also refers to a set of functions
- $\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq cg(n) \leq f(n) \ \forall \ n \geq n_0\}$. Use Ω-notation to give lower bound on a function, to within a constant factor
- Note that $f(n) = \Theta(g(n))$ also implies $f(n) = \Omega(g(n))$
  - Θ-notation is stronger than Ω-notation
- Given a function $g(n)$, we denote as $\Omega(g(n))$ to the set of functions such that:
- For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
- If we say the running time of an algorithm is $\Omega(g(n))$, we mean:
  - No matter what particular input of size n is chosen for each value of n, the running time on that input is at least a constant times $g(n)$ for sufficiently large n
  - We can also mean best-case times, as in insertion sort is $\Omega(n)$.
- Insertion Sort belongs to both, $\Omega(n)$ and $O(n^2)$
- In $n = O(n^2)$, the $=$ sign means set membership: $n \in O(n^2)$
- When we use asymptotic notation in a formula, we interpret it as representing a function for which we don't care about its name, only its order
  - In $2n^2+3n+1 = 2n^2+\Theta(n)$ means $2n^2+3n+1 = 2n^2 + f(n)$, where $f(n)$ is a function in the set $\Theta(n)$
- Example: $2n^2 + 3n + 1 = 2n^2 + \Theta(n) = \Theta(n^2)$.

Asymptotic Notation in Equations and Inequalities

- Given a function $g(n)$, we denote as $O(g(n))$ to the set of functions such that:
- We often describe the running time of an algorithm by inspecting the algorithm's overall structure
  - We use O-notation
  - A doubly nested loop structure shows an $O(n^2)$ upper bound on the worst case running time

o-notation

- The upper bound O-notation may or may not be asymptotically tight
- $2n^2 = O(n^2)$ is asymptotically tight
- $2n = O(n^2)$ is not asymptotically tight
- We use $o - notation$ to denote an upper bound that is not asymptotically tight
- We define $o(g(n))$ and say little-oh of g of n as the set $o(g(n)) = \{f(n)$: for any positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n) \ \forall \ n \geq n_0\}$. Example, $2n = o(n^2)$ but $2n^2 \neq o(n^2)$.

ω-notation

- ω-notation is to Ω-notation as o-notation is to O-notation
  - Use ω-notation to denote a lower bound that is not asymptotically tight
  - $f(n) \in \omega(g(n))$ if and only if $g(n) \in o(f(n))$.
- Formally
  - $\omega(g(n)) = \{f(n)$: for any positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq cg(n) < f(n) \ \forall \ n \geq n_0\}$.
- $n^2/2 = \omega(n)$
- $n^2/2 \neq \omega(n^2)$

Comparing functions

- Many relational properties of real numbers also apply to asymptotic comparisons
- Properties of Asymptotic Functions: Transitivity, Reflexive, symmetry, Transpose Symmetry
- Analogy between asymptotic comparison of 2 functions f and g and comparison of 2 real numbers a and b:
- Trichotomy: for any 2 real numbers a and b, exactly one of the following must hold: $a < b$, $a = b$, or $a > b$
  - NOT all functions are asymptotically comparable
  - Given 2 functions $f(n)$ and $g(n)$, it could be that neither $f(n) = O(g(n))$ nor $f(n) = \Omega(g(n))$ holds