Lecture 6: Sorting Part B

Sorting in Linear Time

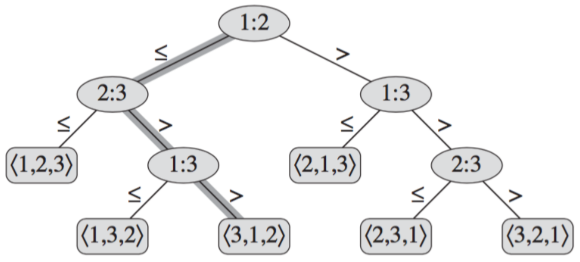
- Previous sorting algorithms
    - Were based on comparisons, comparison sort algorithms
    - Can sort n numbers in $O(n \lg n)$ time
        - Merge sort and heapsort achieve this upper bound in the worst case
        - Quicksort achieves this upper bound in the average case

Lower Bounds for Sorting

- Comparison sort algorithms only use comparisons to get information about an input sequence $\langle a_1, a_2, \ldots, a_n \rangle$
    - One test: $a_i < a_j$, $a_i \leq a_j$, $a_i = a_j$, $a_i \geq a_j$, $a_i > a_j$, to determine order
- For the algorithms in this section
    - Assume all input elements are distinct
    - Comparisons $a_i = a_j$ are useless
    - We only use comparisons of form $a_i \leq a_j$

The Decision-tree Model

- Can view comparison sorts as decision trees
    - Decision tree, a full binary tree representing the comparisons between elements performed by a particular sorting algorithm, given an input
    - Control, data movement, other aspects of algorithm
        - Ignored in the decision tree



The decision tree for insertion sort operating on three elements. An internal node annotated by $i : j$ indicates a comparison between a_i and a_j. A leaf annotated by the permutation $\rangle\pi(1), \pi(2), \ldots, \pi(n)\langle$ indicates the ordering $a_\pi(1) \leq a_\pi(2) \leq \ldots \leq a_\pi(n)$. The shaded path indicates the decisions made when sorting the input sequence $\langle a_1 = 6, a_2 = 8, a_3 = 5\rangle$ the permutation $\langle 3, 1, 2\rangle$ at the leaf indicates that the sorted ordering is $a_3 = 5 \leq a_1 = 6 \leq a_2 = 8$. There are $3! = 6$ possible permutations of the input elements, and so the decision tree must have at least 6 leaves.

- Decision-tree Model
    - Internal nodes annotated with $i : j$, $1 \leq i$, $j \leq n$, n is the number of elements in the input sequence
    - Leaf nodes annotated by a permutation $\langle\pi(1), \pi(2), \ldots, \pi(n)\rangle$
- Execution of the sorting algorithm
    - Tracing a simple path from root down to leaf
    - Each internal node indicates a comparison $a_i \leq a_j$
    - Left subtree, $a_i \leq a_j$
    - Right subtree, $a_i > a_j$
    - When we reach a leaf, we have found the ordering of the elements $a_{\pi(1)} \leq a_{\pi(2)} \leq \cdots \leq a_{\pi(n)}$
    - Each of the $n!$ permutations must appear as a leaf for comparison sort to be correct
    - Each leaf (permutation) must be reachable from the root node
    - Theorem 8.1
        - Any comparison sort algorithm requires $\Omega(n \lg n)$ comparisons in the worst case.
    - Why?
        - Has to do with the height of the decision tree
        - Permutaions (defining ordering) appear as reachable leafs
    - Corolary 8.2
        - Heapsort and merge sort are asymptotically optimal comparison sorts

Counting Sort

- Assume input of n integer elements in a range from 0 to k
- When $k = O(n)$, sorting runs in $\Theta(n)$
- Determine for each input element x, the number of elements smaller than x
  - This is how it positions x in its place in the output array
  - If there are 17 elements smaller than $x \rightarrow x$ will be assigned position 18
  - Must modify scheme in order to deal with repeated numbers

  - $A[j]$, element of the original array
  - $C[A[j]]$, number of repetitions of number in $A[j]$
  - $B[C[A[j]]]$, final array, puts the number in its final position

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COUNTING-SORT(A, B, k)

 1. let $C[0 \dots k]$ be a new array
 2. for $i = 0$ to $k$
 3.     $C[i] = 0$
 4. for $j = 1$ to $A.length$
 5.     $C[A[j]] = C[A[j]] + 1$
 6. // $C[i]$ now contains the number of elements less than or equal to $i$
 7. for $i = 1$ to $k$
 8.     $C[i] = C[i] + C[i-1]$
 9. // $C[i]$ now contains the number of elements less than or equal to $i$
10. for $j = A.length$ downto 1
11.     $B[C[A[j]]] = A[j]$
12.     $C[A[j]] = C[A[j]] - 1$
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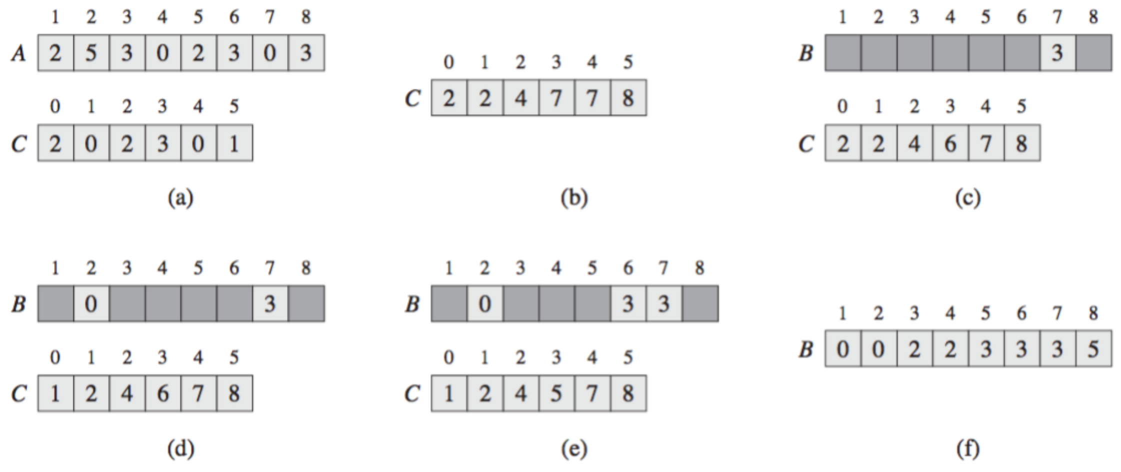
- Lines 2-3 initialize array C to zeros, $\Theta(k)$
- Lines 4-5, sets $C[i]$ to the number of elements equal to i, $\Theta(n)$
- Lines 7-8, determine for each $i = 0, 1, \dots, k$, how many input elements are less than or equal to i, keeping a running sum of the array C, $\Theta(k)$
- Lines 10-12, places each element $A[j]$ into its correct sorted position in B, $\Theta(n)$
- Counting sort beats the lower bound of $\Omega(n \lg n)$
  - It is not a comparison sort
  - It uses the values of the elements to index into an array
  - $\Omega(n \lg n)$ doesn?t apply, it is not a comparison sort model

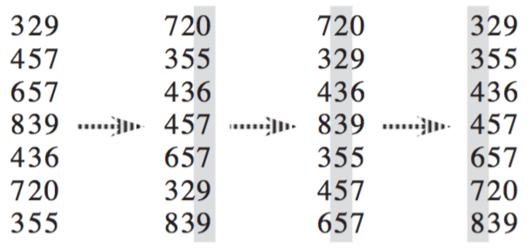
- Important property
  - Counting sort is stable
  - Numbers with the same value appear in the output array in the same order as they do in the input array
  - Important when satellite data is moved around with sorted elements
  - Used as subroutine of radix sort, important property for radix sort to be correct

- The operation of COUNTING-SORT on an input array $A[1 \dots 8$, where each element of A is a nonnegative integer no larger than $k = 5$. (a) The array A and the auxiliary array C after line 5. (b) The array C after line 8. (c)–(e) The output array B and the auxiliary array C after one, two, and three iterations of the loop in lines 10–12, respectively. Only the lightly shaded elements of array B have been filled in. (f) The final sorted output array B.



Radix Sort

- Radix Sort was originally used by card-sorting machines (IBM)
  - Cards have 80 columns, each column has 12 places, a machine punches one of those holes
  - Card-sorting machines worked with one column at a time
- Currently, radix sort is used for multi-key sorting (i.e. year/month/day)
- Considers each digit of the number as an independent key
- How do we sort?
  - Intuitively, we would sort numbers by the most significant digit, sort each of the resulting bins recursively, then combine the decks in order
- Problem
  - the cards in 9 of the 10 bns must be put aside to sort each of the bins
  - this procedure generates many intermediate piles of cards
  - We would have to keep track of these piles
- How do we sort?
  - Radix sort works from the least significant digit first
  - It combines the cards into a single deck
  - Cards in the 0 bin precede cards in the 1 bin which precede the cards in the 2 bin, and so on
  - Then sorts the entire deck again on second least significant digit and recombines the deck
  - Process continues until cards have been sorted on all d digits
- Radix sort requires d passes



The operation of radix sort on a list of seven 3-digit numbers. The leftmost column is the input. The remaining columns show the list after successive sorts on increasingly significant digit positions. Shading indicates the digit position sorted on to produce each list from the previous one.

- d is the number of digits
  - Digit 1 is has lowest order
  - Digit d has the highest order

RADIX-SORT(A, d)

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1. for $i = 1$ to $d$
2.       use a stable sort to sort array $A$ on digit $i$
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- Analysis of Radix Sort
  - If each digit is in the range from 1 to k, we use CountingSort
  - Each step for a digit takes $\Theta(n + k)$
  - For d digits $\Theta(dn + dk)$
  - If d is a constant and $k = O(n)$, $T(n) = O(n)$
  - Radix-n means that each digit can differentiate among n different symbols, in previous examples we used radix-10 (digits from 0 to 9).

Bucket Sort

- Bucket sort runs in linear time when the input elements are drawn from a uniform distribution, average-case running time $O(n)$
- Fast because it assumes something about the input
  - (i.e. Counting sort assumes numbers in a small range)
  - Bucket sort assumes numbers randomly generated and uniformly distributed in a range $[0, 1)$
- Divides the interval $[0, 1)$ in n sub-intervals (or buckets) of the same size
  - Numbers distributed in buckets
  - Don?t expect many numbers to fall in each bucket (they are uniformly and independently distributed over $[0, 1)$)
  - To sort the numbers, we sort numbers in each bucket, then go through buckets in order

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BUCKET SORT($A$)
1. let $B[0 \ldots n-1$ be a new array
2. $n = A.length$
3. for $i = 0$ to $n-1$
4. make $B[i]$ an empty list
5. for $i = 1$ to $n$
6. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
7. for $i = 0$ to $n-1$
8.     sort list $B[i]$ with insertion sort
9. concatenate the lists $B[0], B[1], \ldots B[n-1]$ together in order
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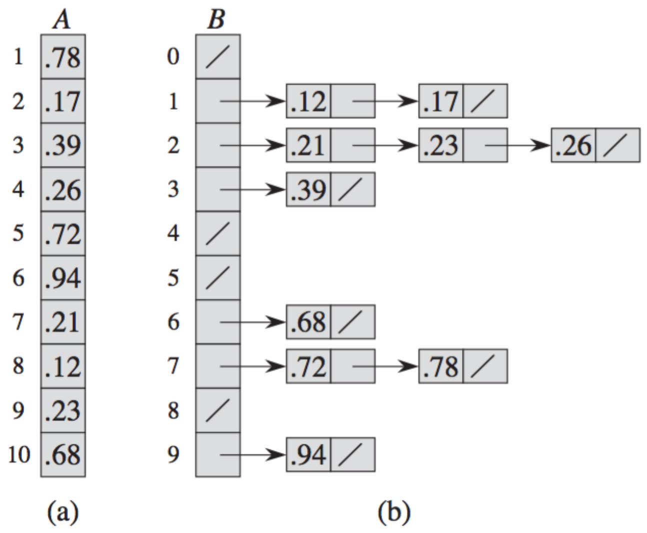

The operation of BUCKET-SORT for $n = 0$. (a) The input array $A[1 \ldots 10]$. (b) The array $B[0 \ldots 9]$ of sorted lists (buckets) after line 8 of the algorithm. Bucket i holds values in the half-open interval $[i/10, (i+1)/10)$. The sorted output consists of a concatenation in order of the lists $B[0], B[1], \ldots B[9]$