Computational Geometry

TOPICS

- Preliminaries
- Point in a Polygon
- Polygon Construction
- Convex Hulls

Geometric Algorithms

Geometric Algorithms find applications in such areas as

- Computer Graphics
- Computer Aided Design
- VLSI Design
- GIS
- Robotics

algorithms dealing with

points, lines, line segments, and polygons

In particular, the algorithms will

- Determine whether a point is inside a Polygon
- Construct a Polygon
- Determine Convex Hulls

Preliminaries:

A **point** p is represented as a pair of coordinates (x,y)
A **line** is represented by a pair of points
A **path** is a sequence of points $p_1,p_2, \ldots p_n$ and the line segments connecting them,

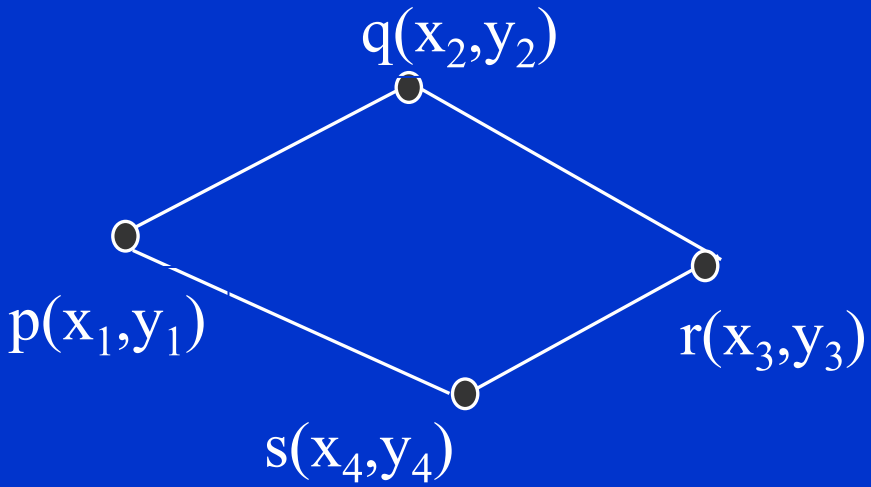
$$p_1\text{-}p_2,\ p_2\text{-}p_3,\ \ldots,\ p_{k-1}\text{-}p_k.$$

A **closed path** whose last point is the same as the first is a polygon.

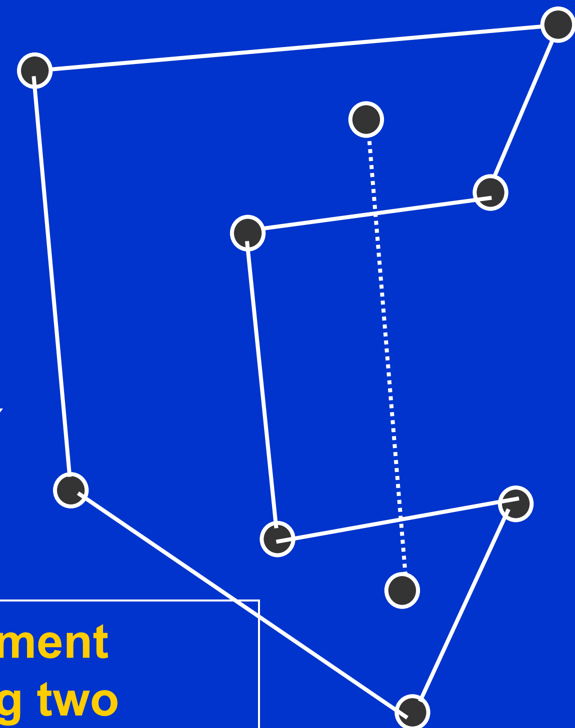
A **simple polygon** is one whose corresponding path does not intersect itself. It encloses a region in the plane.

A **Convex Polygon** is a polygon such that any line segment connecting two points inside the polygon is itself entirely in the polygon.

The **convex hull** of a set of points is defined as the smallest convex polygon enclosing all the given points.

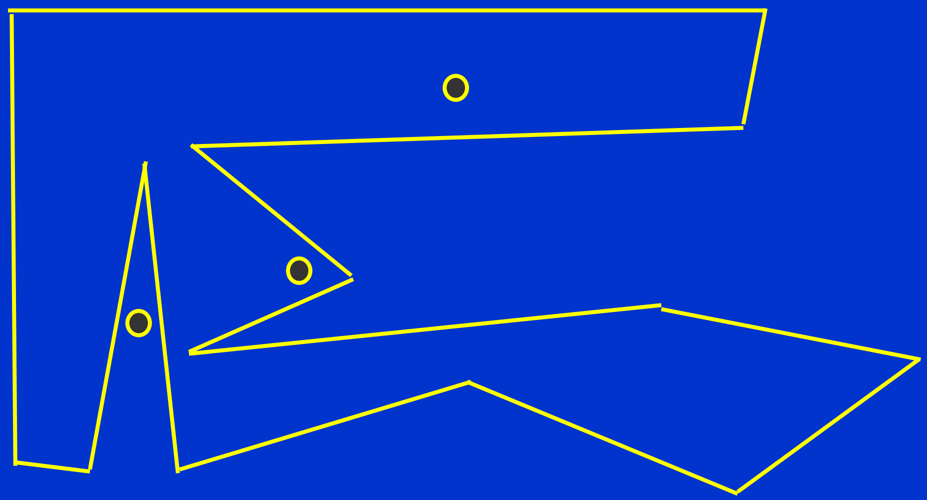


This is not a convex polygon



A line segment connecting two points:
The points are inside the polygon
The line segment is not entirely in the polygon

Determining whether a *point* is inside a polygon



Given a simple polygon P, and a point q, determine whether the point is inside or outside the polygon. (a non-convex polygon)

Procedure **Point_in_a_Polygon(P,q)**

Input : P (a simple polygon with vertices p_1,p_2,p_3, and edges e_1,e_2,e_3, … e_n and q (x_0,y_0) a point.

Output: INSIDE (a Boolean variable, True if q is inside P, and false otherwise)

Count $\leftarrow$ 0;

for all edges e_i of the polygon do

if the line $x = x_0$ intersects e_i then

$y_i \leftarrow$ y coordinate of the intersection between lines e_i and $x=x_0$;

if $y_i > y_0$ **then**

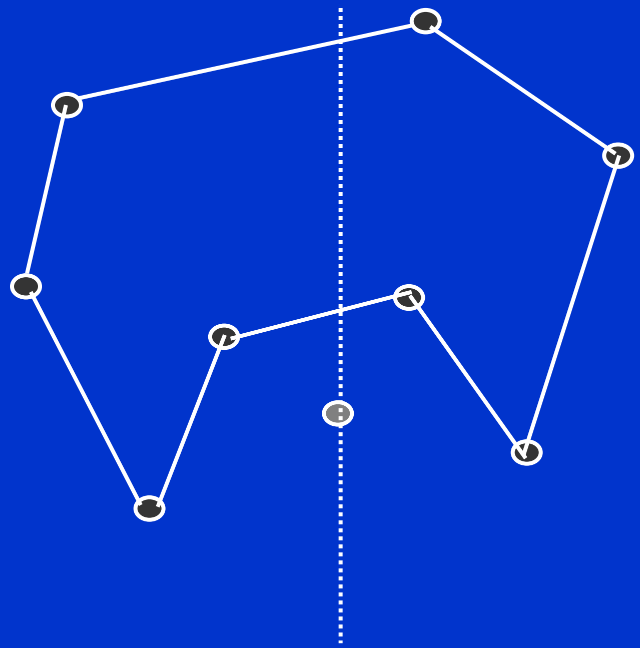
Count $\leftarrow$ Count +1;

if count is odd **then** INSIDE $\leftarrow$ TRUE;

else INSIDE $\leftarrow$ FALSE

This does not work if the line passes through terminal points of edges

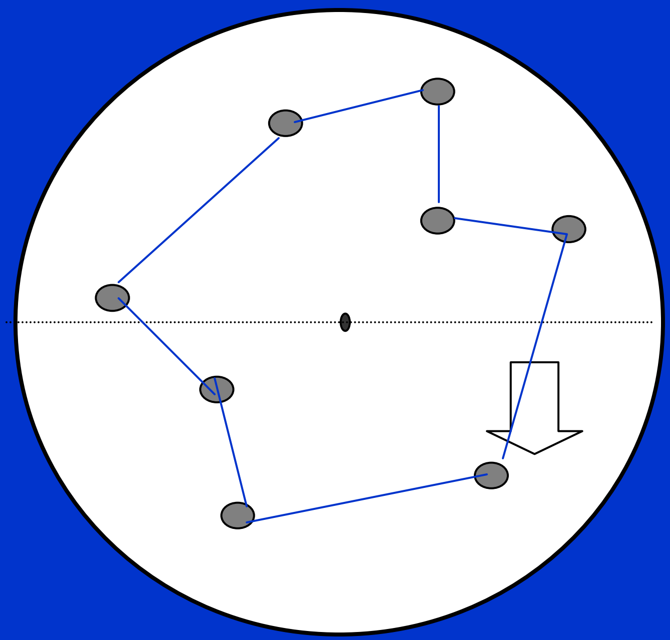
It takes constant time to perform an intersection between two line segments.
The algorithm computes n such intersections, where n is the size on the polygon.
Total running time of the algorithm, O(n).



Constructing a Simple Polygon

Given a set of points in the plane, connect them in a simple closed path.

Consider a large circle that contains all the points.
Scan the area of C by a rotating line.
Connect the points in the order they are encountered in the scan.



Procedure **Simple_Polygon**

Input : $p_1, p_2, \ldots p_n$ (points in the polygon)
Output : P (a simple polygon whose vertices $p_1, p_2, \ldots p_n$ are in some order)

1. **for** $i \leftarrow 2$ **to** n
2. $\alpha_i \leftarrow$ angle between line p_1-p_i and the x-axis;
3. sort the points according to the angles
   (use the corresponding priority for the point and do a heapsort)
4. P is the polygon defined by the list of points in the sorted order.

Complexity : Complexity of the sorting algorithm.

Convex Hulls

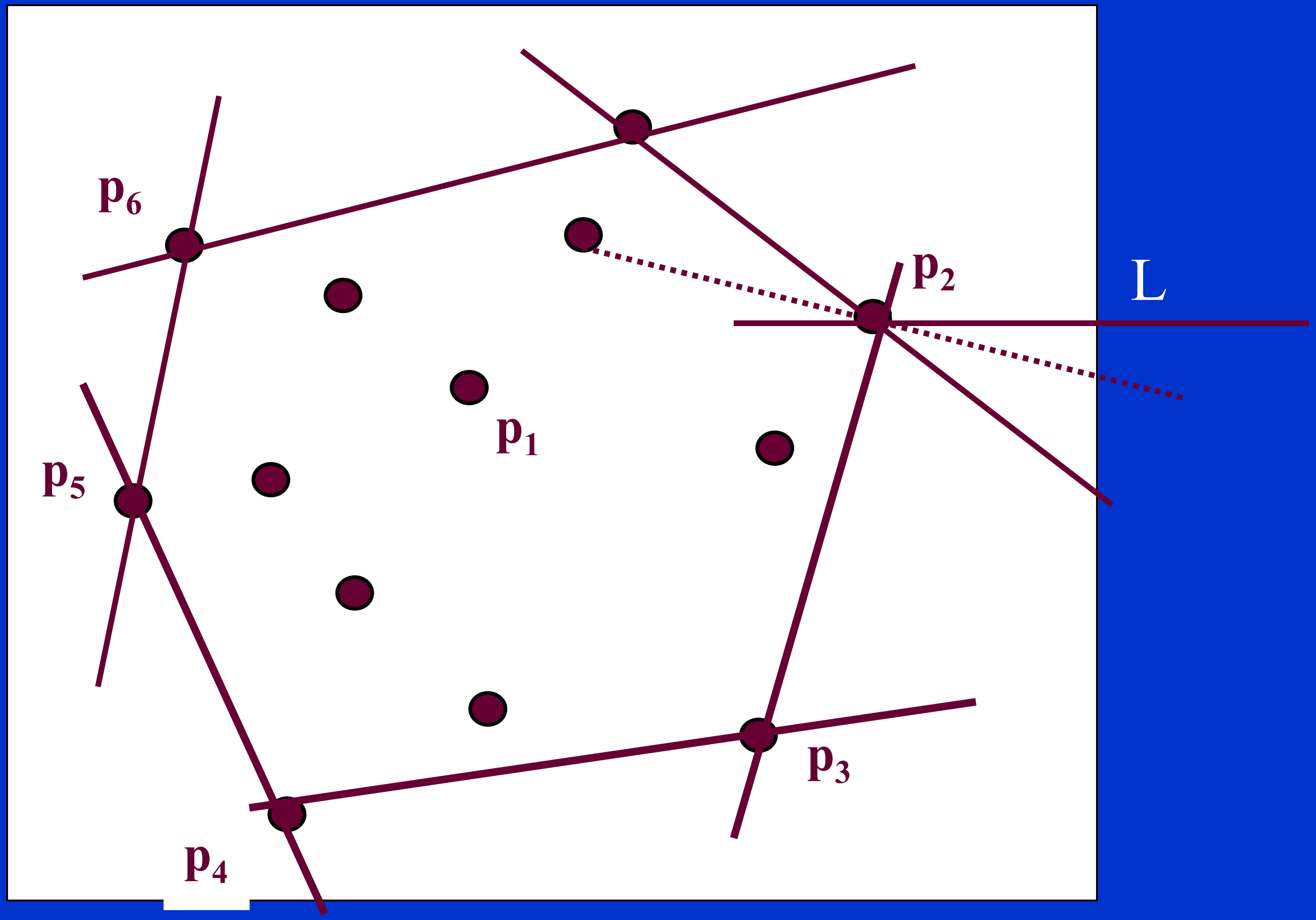
The convex hull of a set of points is defined as the smallest convex polygon enclosing all the points in the set.

The convex hull is the smallest region encompassing a set of points.
A convex hull can contain as little as three and as many as all the points as vertices.

Problem Statement : Compute the convex hull of n given points in the plane.

There are two algorithms

Gift Wrapping $O(n^2)$
Graham's Scan $O(n\log n)$



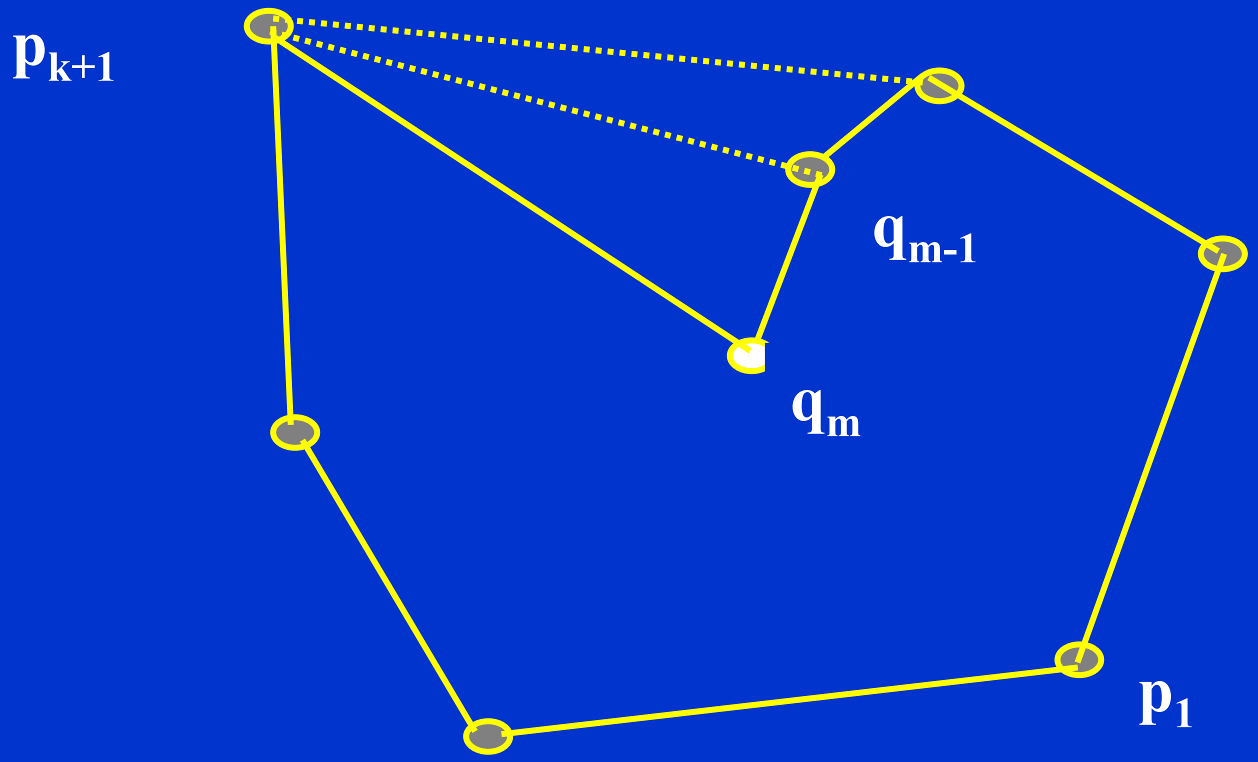
p_6
p_2
L
p_1
p_5
p_3
p_4

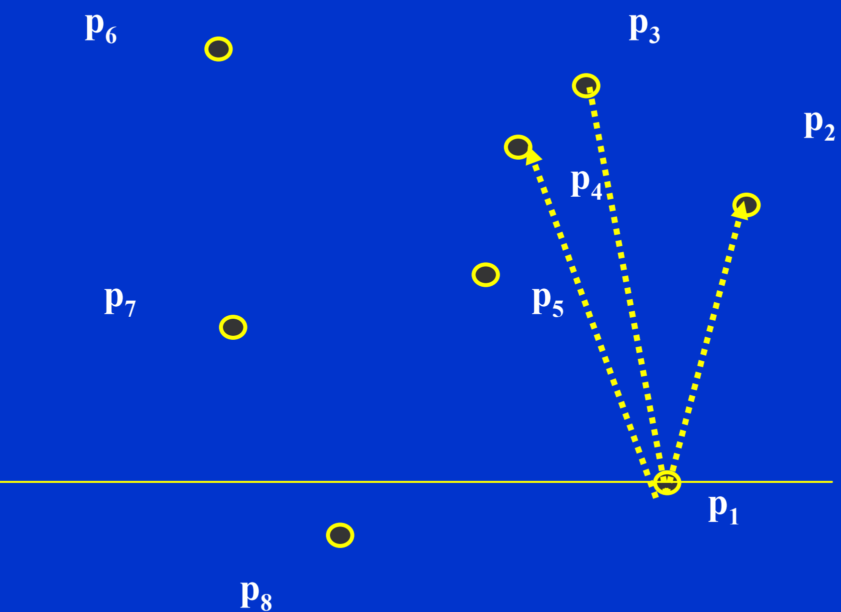
Procedure Gift_Wrapping($p_1, p_2, \ldots p_n$)
Input : $p_1, p_2, \ldots p_n$ (a set of points in the plane)
Output : P (the convex hull of $p_1, p_2, \ldots p_n$)

1. P $\leftarrow$ {0} or ε;
2. p $\leftarrow$ a point in the set with the largest x-coordinate;
3. Add p to P;
4. L $\leftarrow$ line containing p and parallel to the x-axis;
5. **while** |P| < n **do**
6. q $\leftarrow$ point such that the angle between the line -p-q- and L is minimal among all points;
7. add q to P;
8. L $\leftarrow$ line -p-q-;
9. p$\leftarrow$q;

Graham's Scan:

Given a set of n points in the plane, ordered as in the algorithm Simple Polygon, we can find a convex path among the first k points whose corresponding convex polygon encloses the first k points.



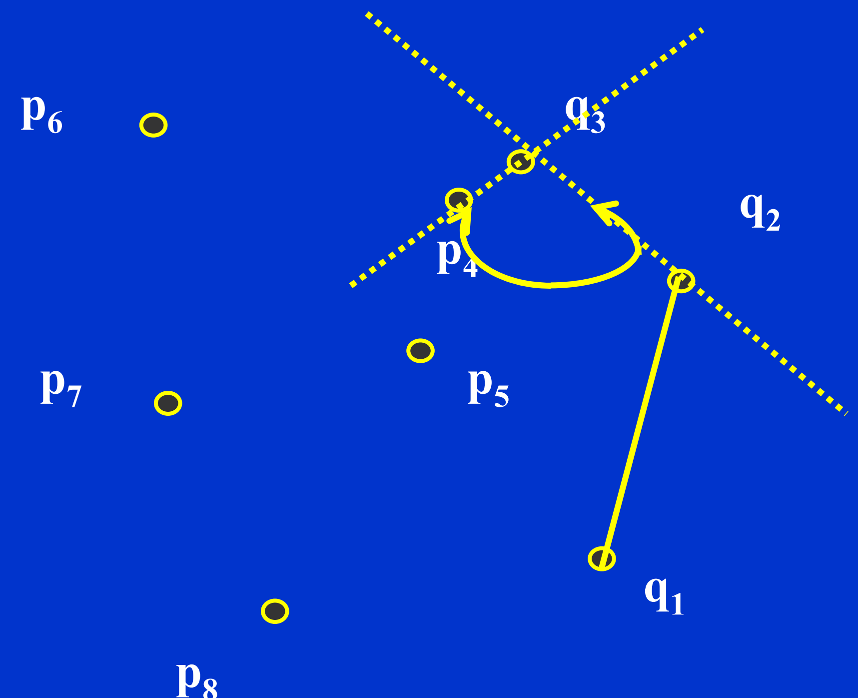


q1:p1

q2:p2

q3:p3

m:3



-q2-q3- and -q3-p4-

The angle is < 180

-q m-1-qm- and -q m-pk - ≥ 180o

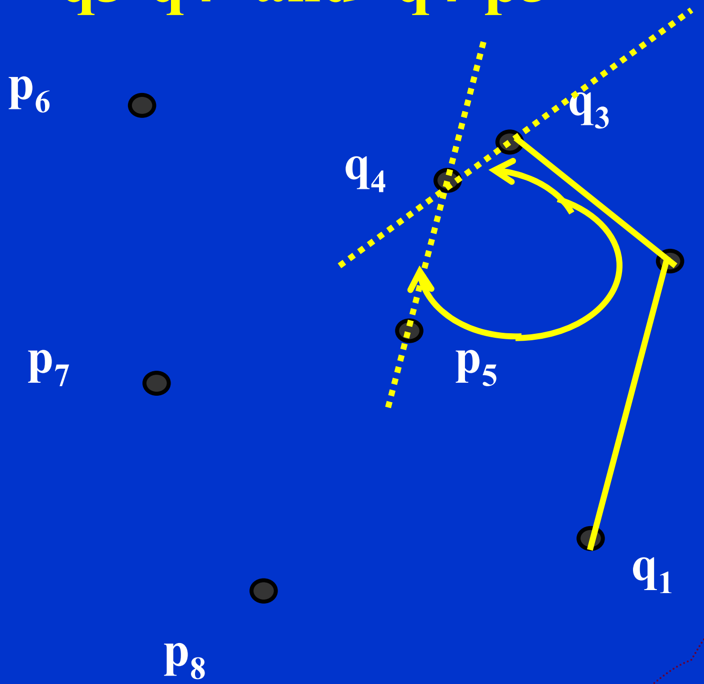
k =4

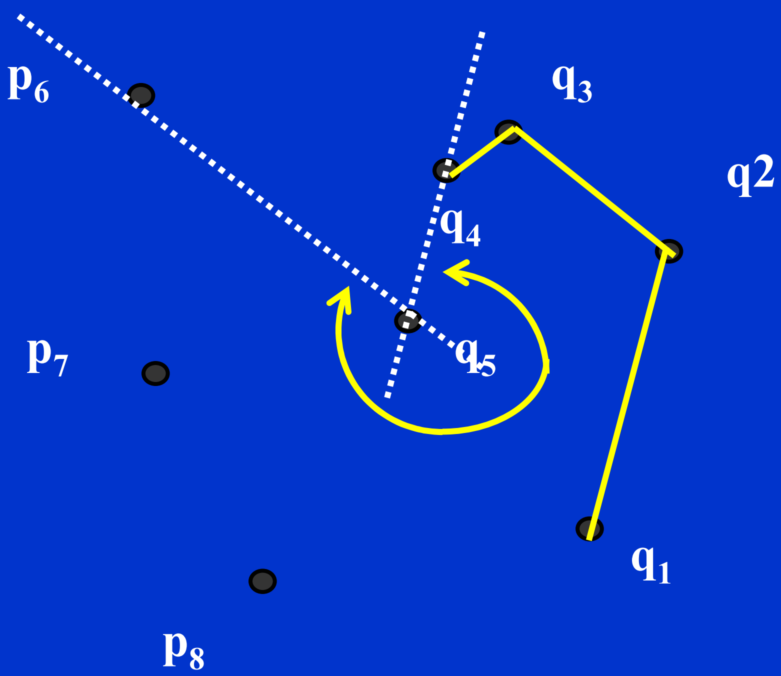
-q3-q4- and -q4-p5-

k =5

q_m:p4

m:4

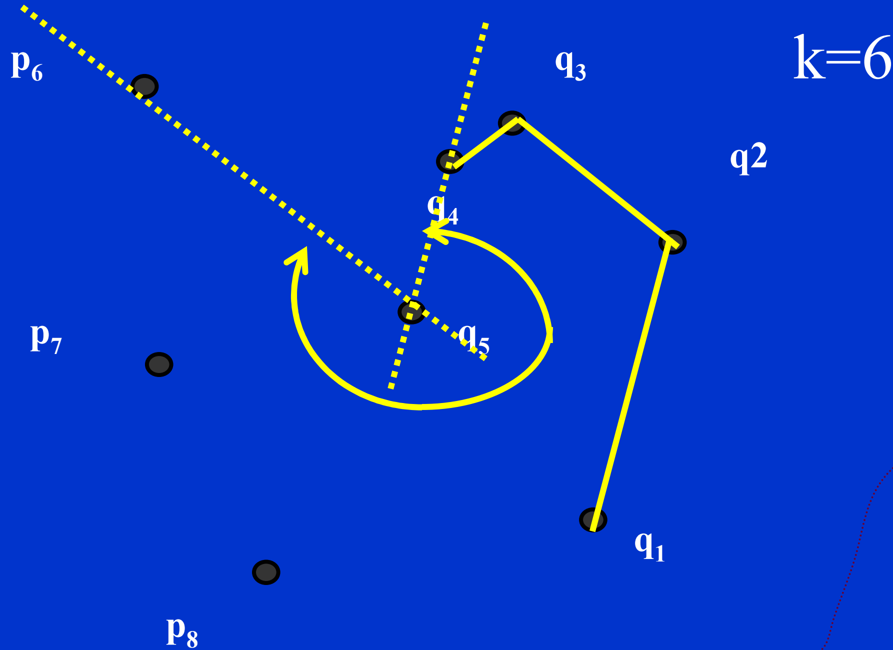




k=6

q_m:p5

m:5

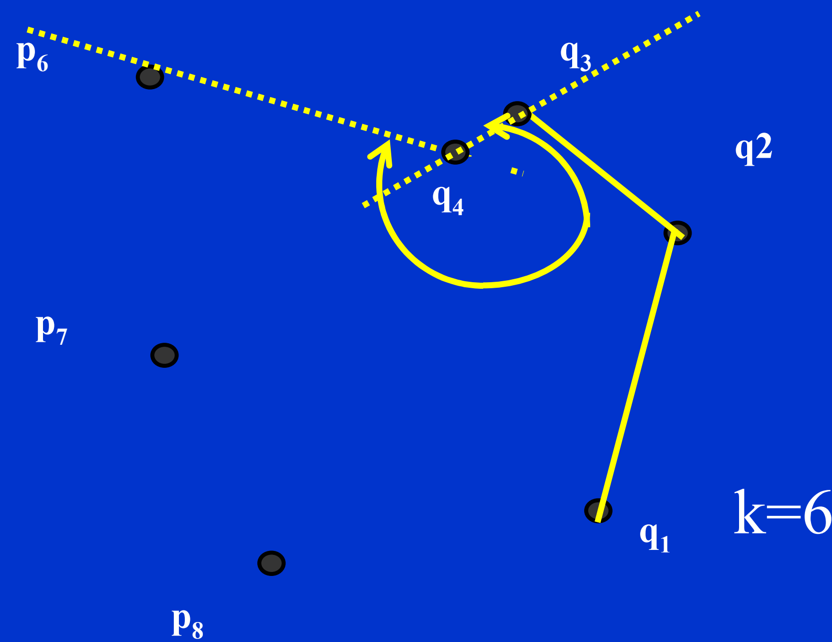


Angle between -q4-q5- and

-q5-p6- is greater than 180

Therefore   m = m-1 = 4

We skip p5



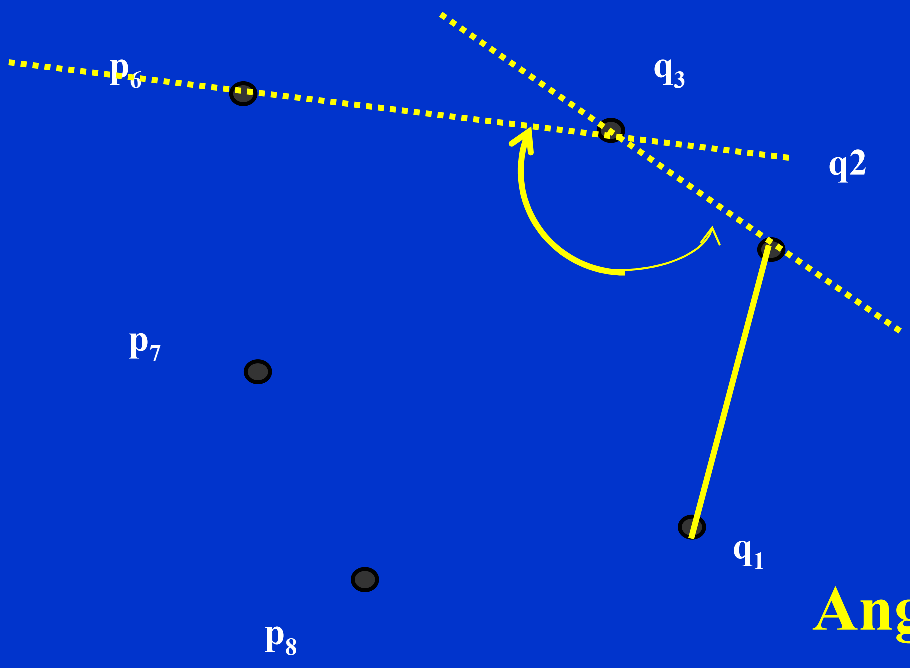
Angle between -q3-q4- and

-q4-p6- is greater than 180

Therefore   m = m-1 = 3

We skip p4

-q3-q4- and -q4-p6-



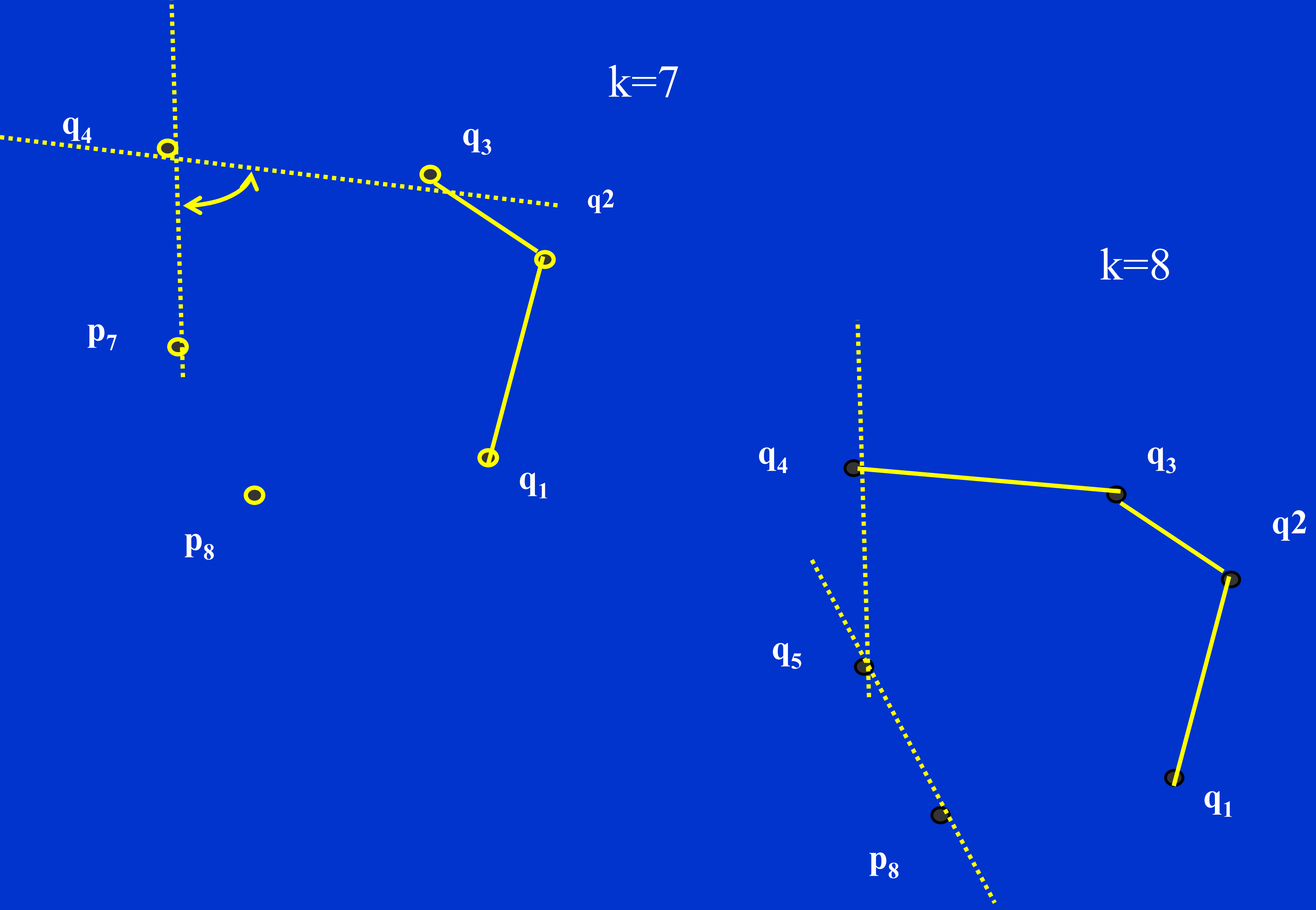
k=6

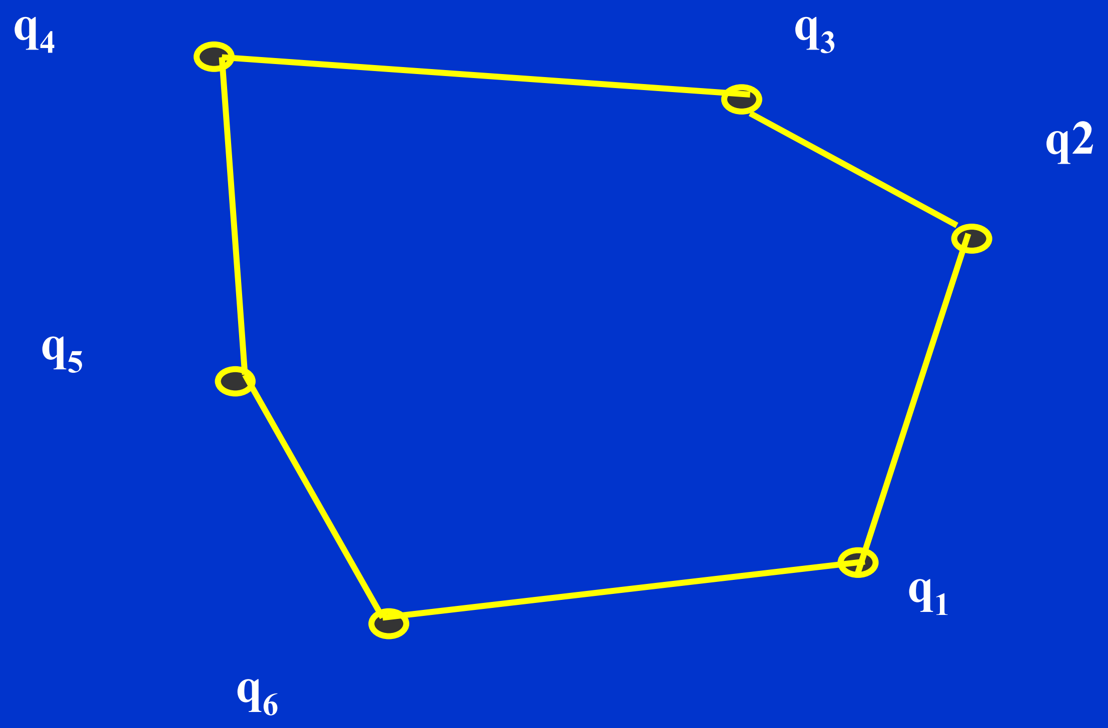
Angle between -q2-q3- and

-q3-p6- is less than 180

Therefore m = m+1 = 4

and q4 = p6


k=7
q4
q3
q2
p7
q1
p8
k=8
q4
q3
q2
q5
q1
p8


q_4
q_3
q2
q_5
q_1
q_6

Procedure Graham's Scan(p_1,p_2, . . . p_n)

Input : p_1,p_2, . . . p_n(a set of points in the plane)

Output : q_1,q_2, . . . q_n(the convex hull of p_1,p_2, . . . p_n)

p1 $\leftarrow$ the point in the set with the largest x-coordinate (and smallest y-coordinate if there are more than one point with the same x-coordinate)

Construct Simple Polygon and arrange points in order

Let order be p_1,p_2, . . . p_n

$q_1 \leftarrow p_1$;

$q_2 \leftarrow p_2$;

$q_3 \leftarrow p_3$;

(initially P consists of p_1,p_2,and p_3)

$m \leftarrow 3$;

for $k \leftarrow 4$ **to** n **do**

while the angle between lines -q_{m-1}-q_m- and -q_m-p_k - $\geq 180^o$ **do**

$m \leftarrow m-1$; **[Internal to the polygon]**

$m \leftarrow m+1$;

$q_m \leftarrow p_k$;