NP-Complete Problems

Chapter 3

Algorithm Design

Kleinberg and Tardos

P and NP

Polynomial time reductions

Satisfiability Problem, Clique Problem, Vertex Cover, and

Dominating Set

Polynomial Algorithms

- Problems encountered so far are polynomial time algorithms
- The worst-case running time of most of these algorithms is $O(n^k)$ time, for some constant k.
- All problems cannot be solved in polynomial time
- There are problems that cannot be solved at all – Unsolvable
- There are problems that can be solved but not in $O(n^k)$ time for some constant time.
- Problems that are solvable in polynomial time by polynomial-time algorithms are said to be tractable (or easy or efficient).
- Problems that require superpolynomial time are said to be intractable or hard.

P and NP

- Class P problems are solvable in polynomial time.
- Class NP problems are verifiable in polynomial time.
  - For example, given a problem, we can verify the solution in polynomial time
- Any problem in P is also in NP
- P $\subseteq$ NP
- It is NOT KNOWN whether P is a proper subset of NP.

NP

- **What are NP-complete Problems**?
  A problem is said to be NP-Complete if it is as ‘hard’ as any problem in NP.
- No polynomial time algorithm has yet been discovered for an NP-Complete problem.
- However, it has not been proven that NO polynomial time algorithm can exist for an NP-Complete problem.
- This problem was first posed by Cook in 1971.
- The issue of, P=NP or P$\neq$ NP is an open research problem

Examples of NP-Complete problems

- Shortest vs. longest simple paths
- Finding the shortest paths from a single source in a directed graph G =(V,E) can be completed in O(V + E) time. Even with negative edge weights.

However, finding the longest simple path between two vertices is NP-complete. It is NP-Complete even if each of edge weights is equal to one.

- **An Euler tour of a connected directed graph G=(V,E), can be completed in O(E) time.**

However, the Hamiltonian Cycle is a NP-Complete problem.

The traveling salesman problem is a variation of the Hamiltonian cycle.

Polynomial Time Reductions

- Decision Problems: Problems whose answer is yes or no!! Most problems can be converted to decision problems.
- Language recognition problem is a decision problem.
- Suppose L $\subseteq$ U is the set of all inputs for which the answer to the problem is yes –
- L is called the language corresponding to the problem (Turing machines).

  The terms language and problem are used interchangeably.
- Given a problem, with an input language X,
- Now the decision problem can be defined as the problem to recognize whether or not X belongs to L.

Reductions Contd.

- Definition: Let L1 and L2 be two languages from input spaces U1 and U2. We say that L1 is polynomially reducible to L2 if there exists a polynomial-time algorithm that converts each input $u1 \in U1$ to another input $u2 \in U2$ such that $u1 \in L1$ iff $u2 \in L2$.
- The algorithm is polynomial in the size of the input u1.
- If we have an algorithm for L2 then we can compose the two algorithms to produce an algorithm for L1.
- If L1 is polynomially reducible to L2 and there is a polynomial-time algorithm for L2, then there is a polynomial time algorithm for L1.
- Reducibility is not symmetric
- L1 is polynomially reducible to L2 does not imply L2 is polynomially reducible to L1

The Satisfiability (SAT) Problem

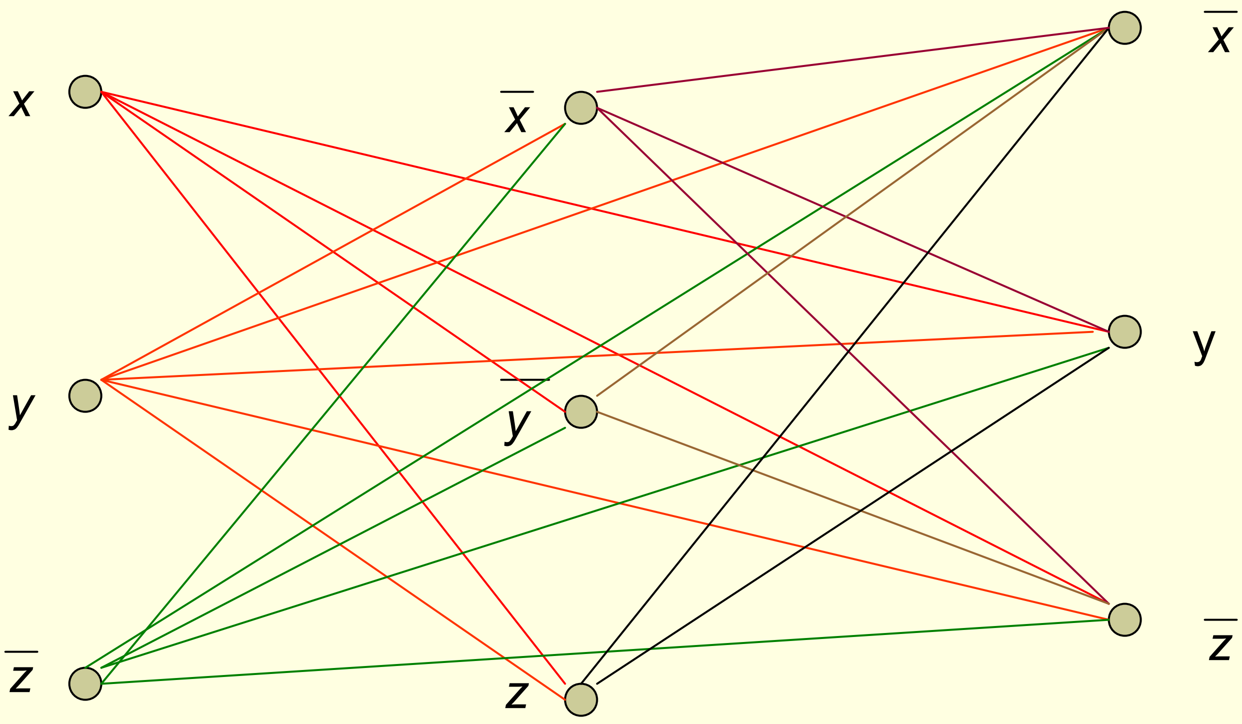
- S – Boolean expression in Conjunctive Normal Form (CNF) (Product (AND) of Sums (ORs))

- For example $S = (x1+x2+\underline{x3})\cdot(\underline{x1}+x2+x3)\cdot(\underline{x1}+\underline{x2}+x3)$
- The SAT problem is to determine whether a given Boolean expression is Satisfiable (without necessarily finding a satisfying assignment)
- We can guess a truth assignment and check that it satisfies the expression in polynomial time.
- SAT is NP hard
- A Turing machine and all of its operations on a given input can be described by a Boolean Expression. The expression will be satisfiable iff the Turing machine will terminate at an accepting state for the given input.

- http://www.nada.kth.se/~viggo/problemlist/compendium.html

Clique Problem

$$(x + y + \overline{z}) \bullet (\overline{x} + \overline{y} + z) \bullet (\overline{x} + y + \overline{z})$$



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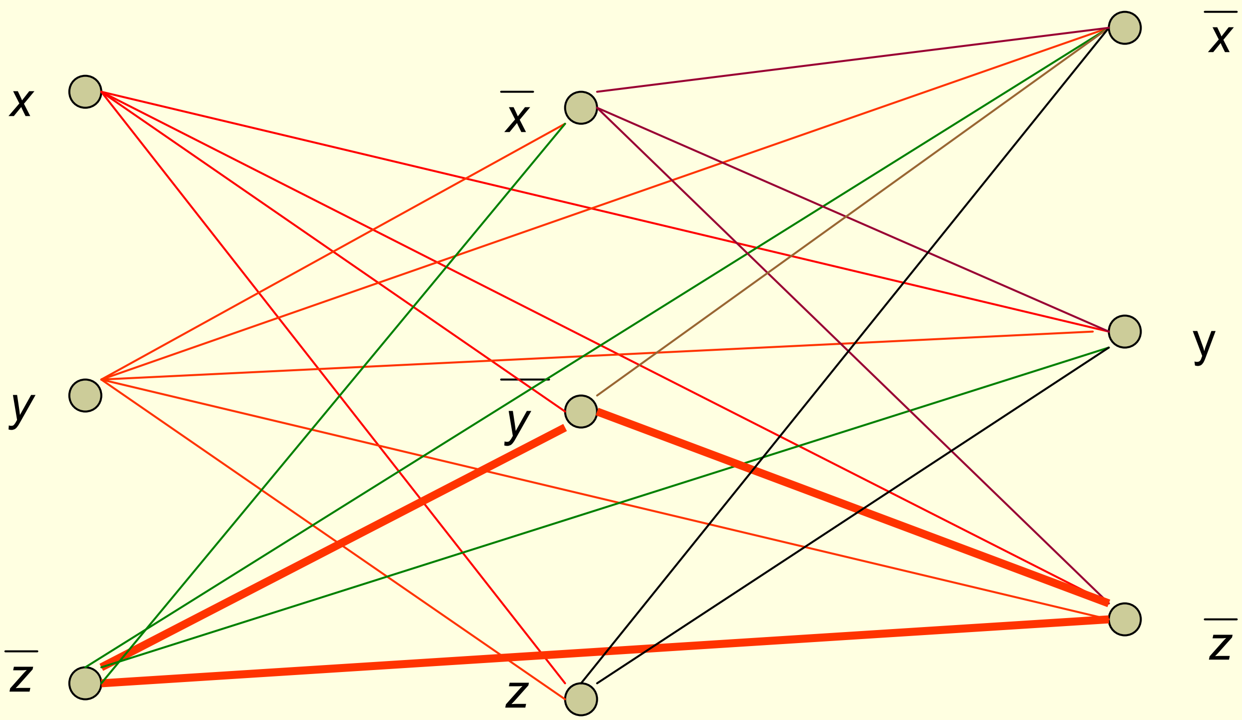

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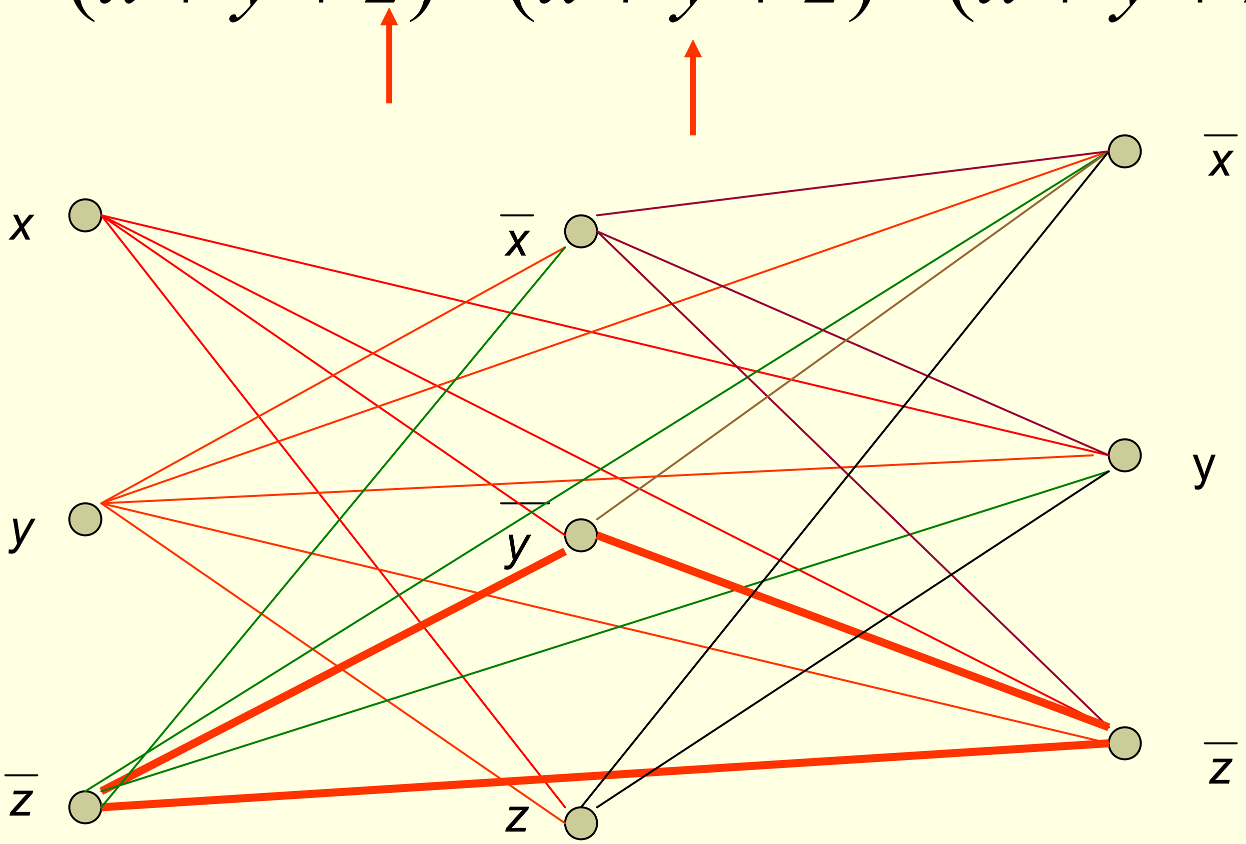

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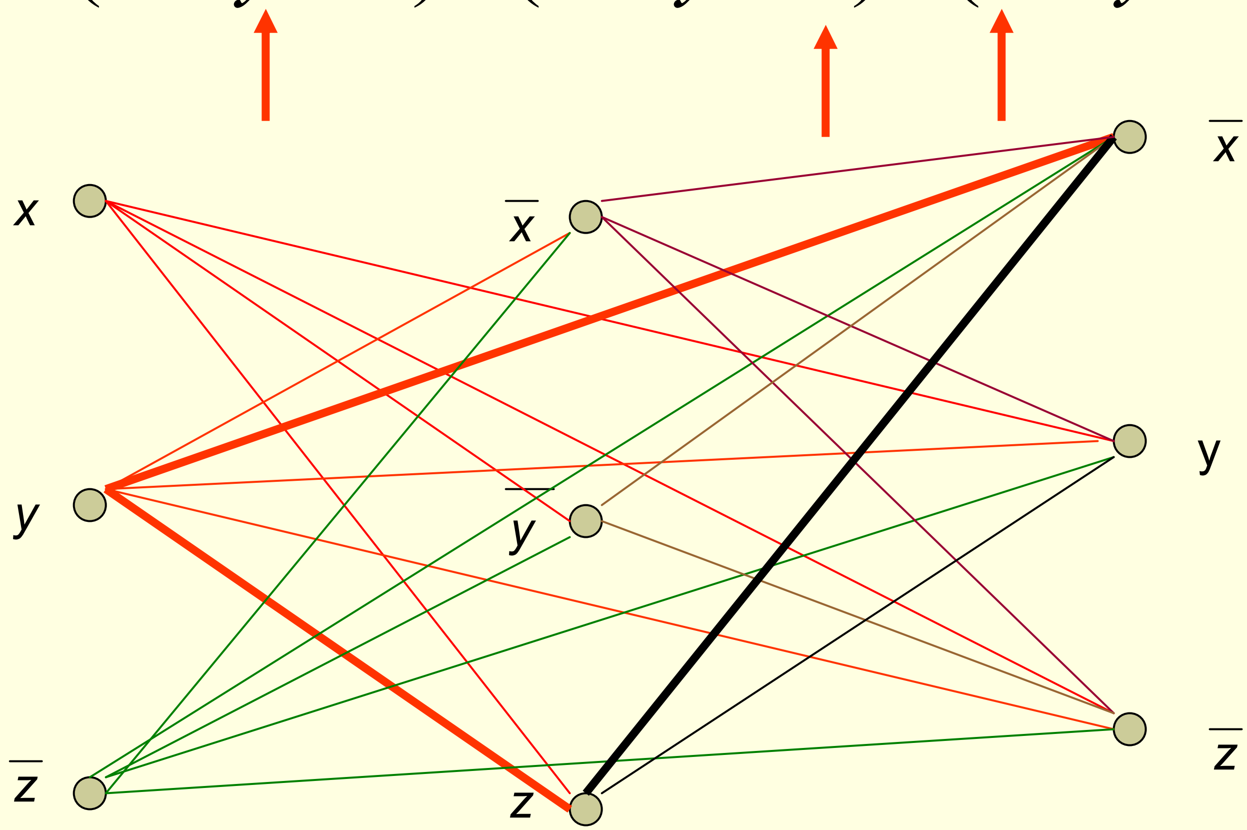

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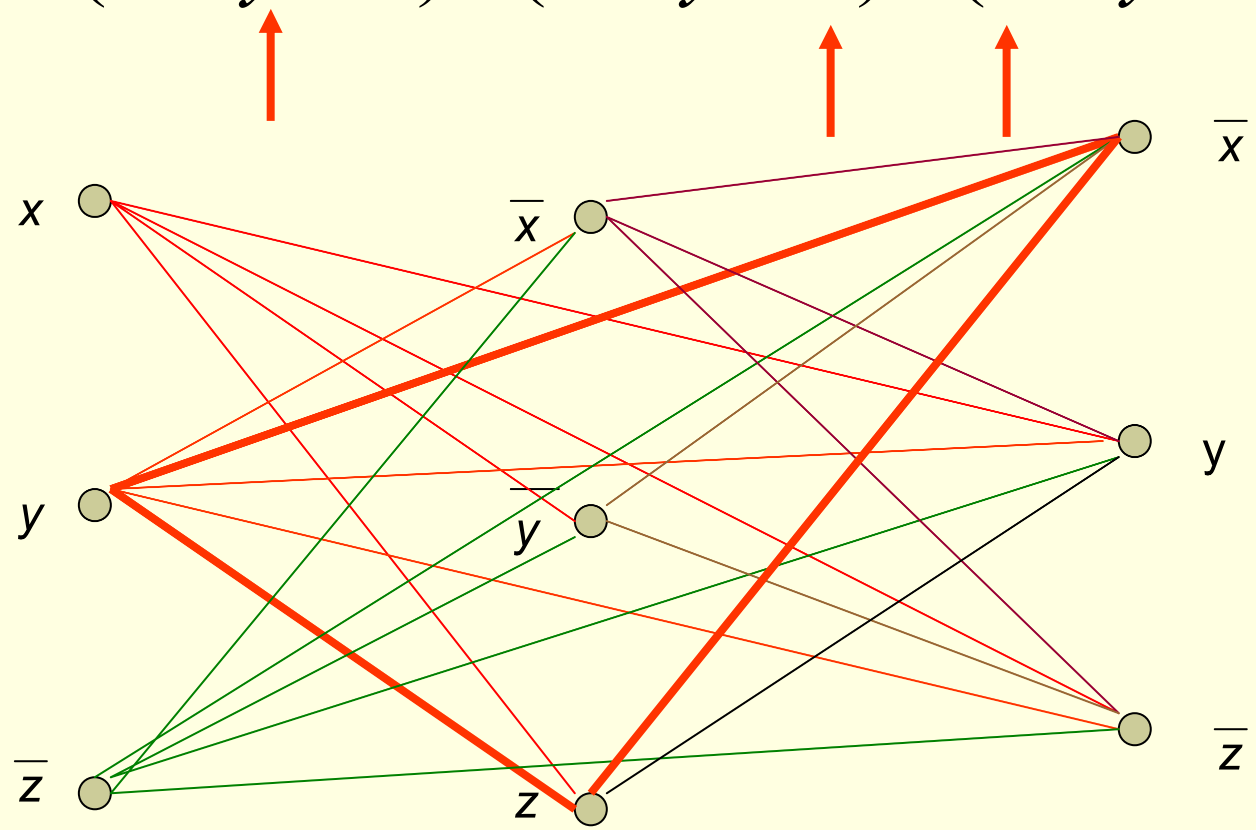
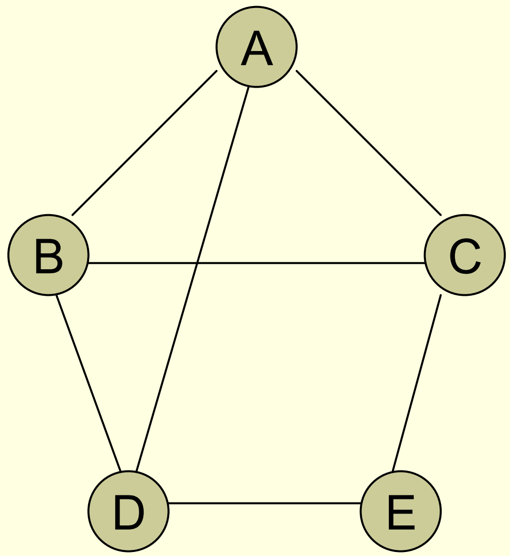


Vertex Cover problem

A vertex cover of G = (V,E) is a set of vertices such that every edge in E is incident to at least one of the vertices in the vertex cover.



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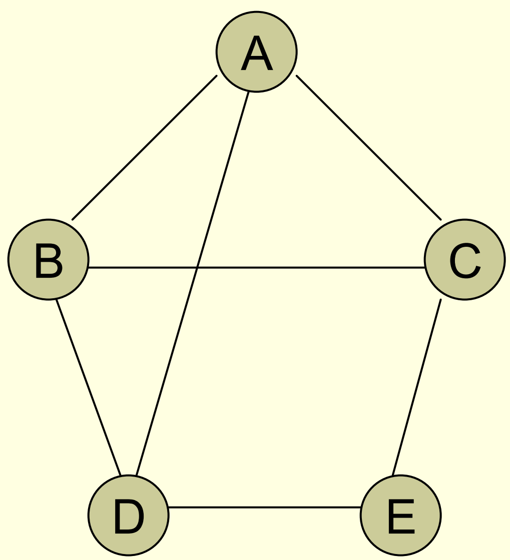
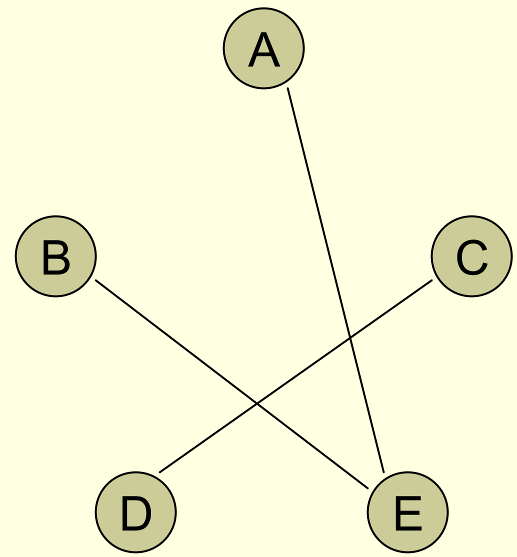


G



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complement of G

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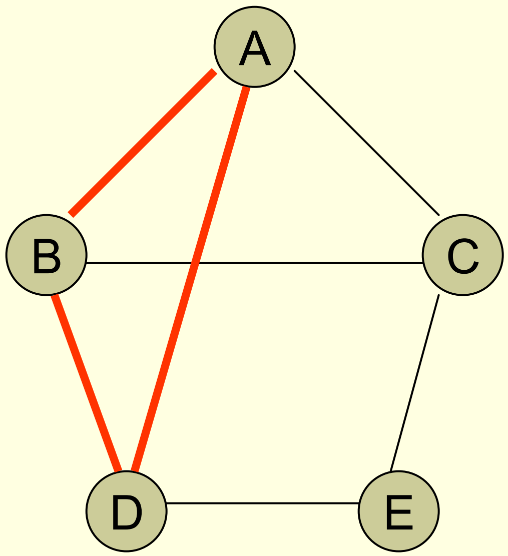

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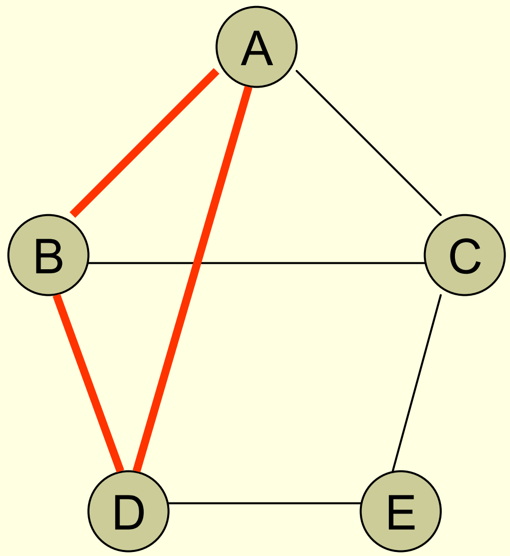

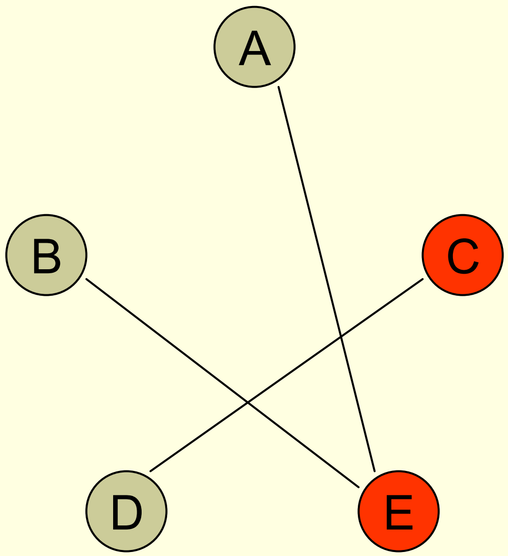

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G' has a minimum vertex cover of size k if and only if G has a maximum clique of size $|V|$-k

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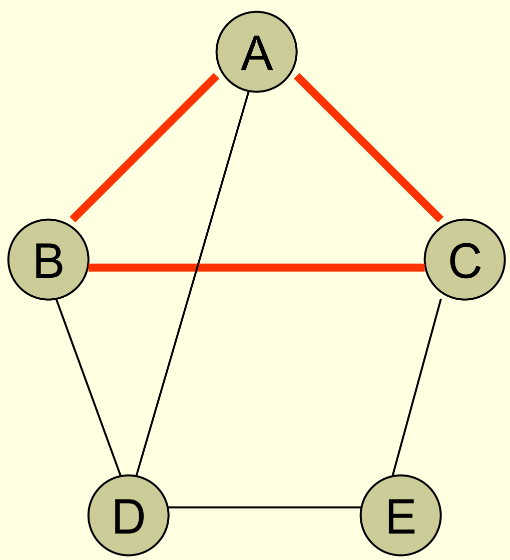
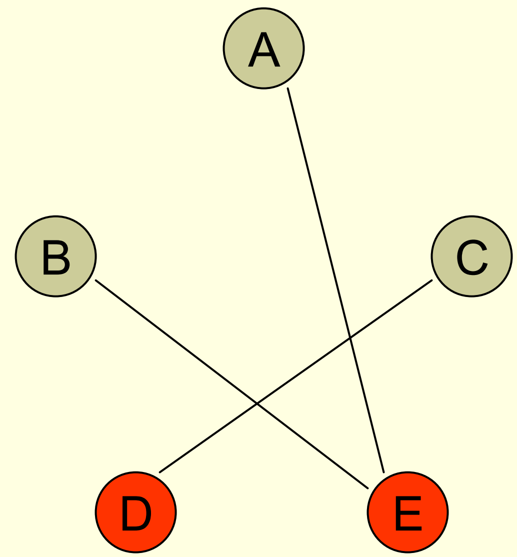


G



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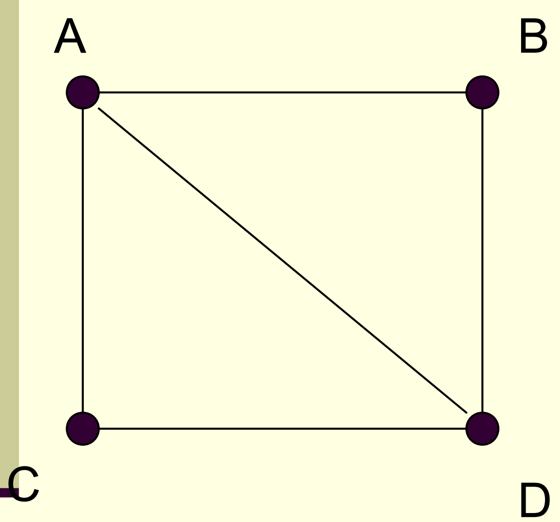
complement of G

Dominating Set

- G = (V,E) is an undirected graph. A dominating set **D** of G is a set of vertices (a subset of V) such that every vertex of G is either in **D** or is adjacent to at least one vertex from **D**.



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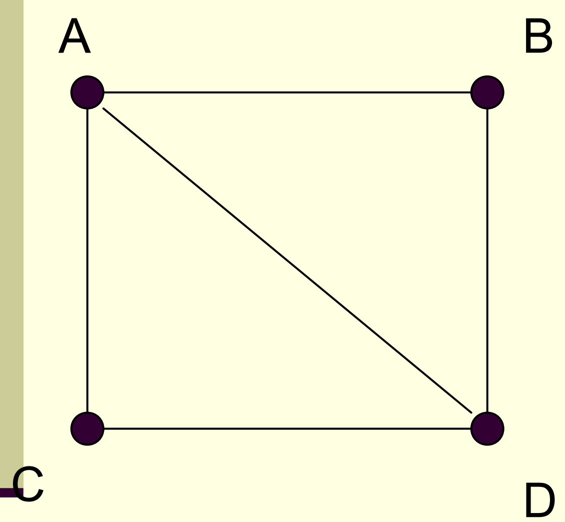

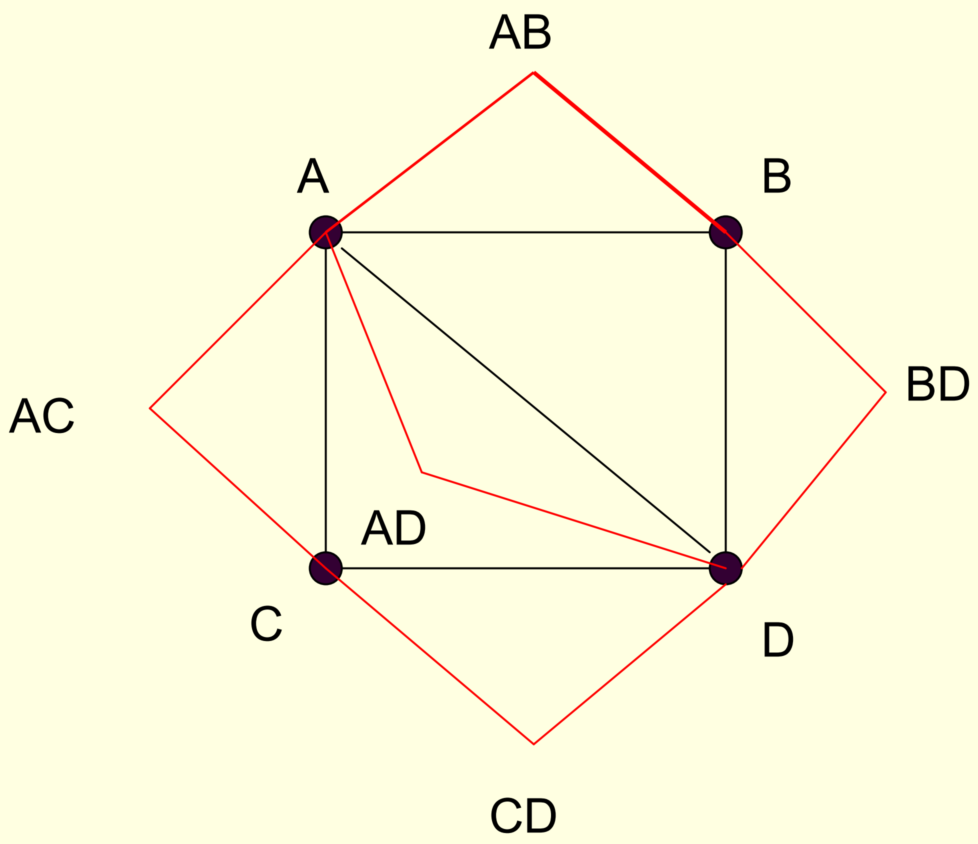

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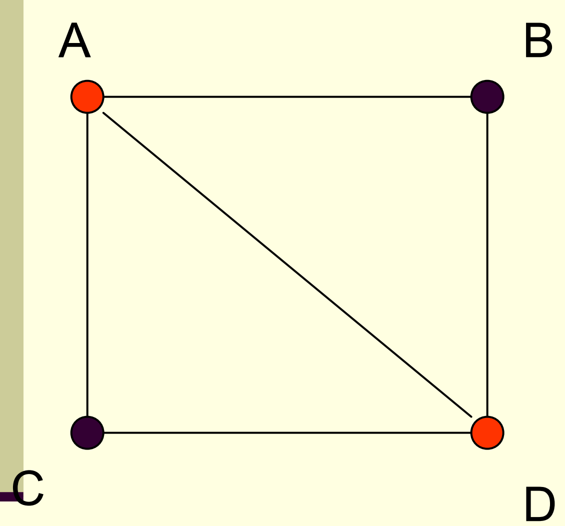

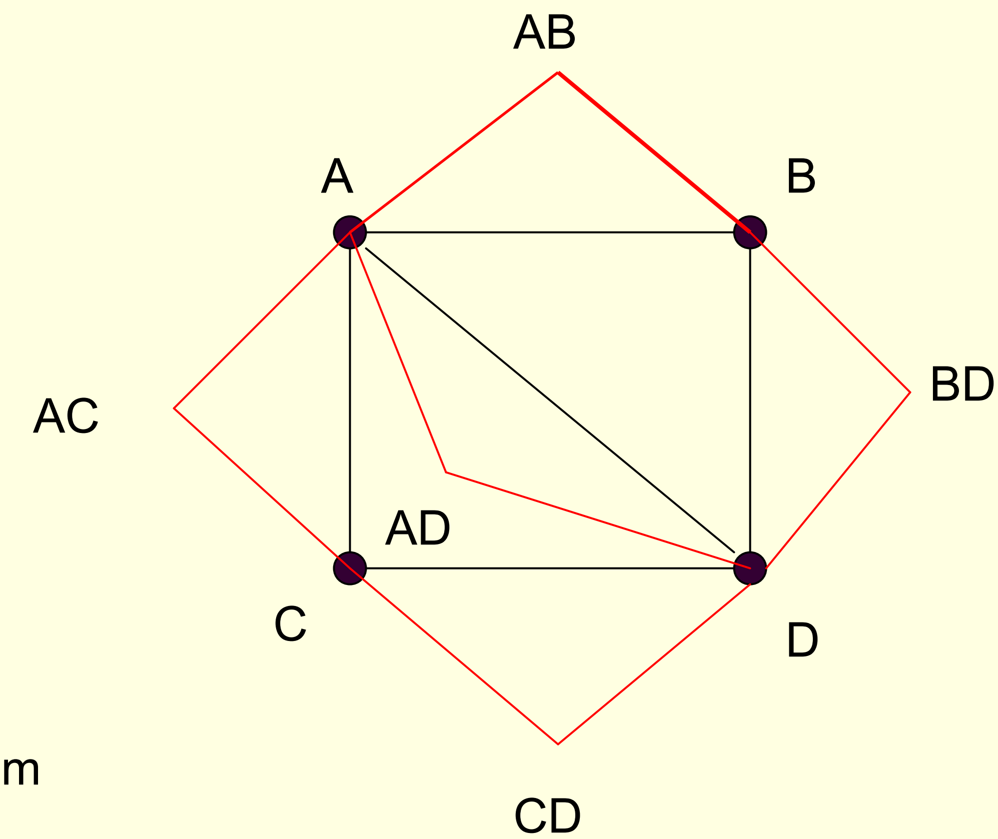

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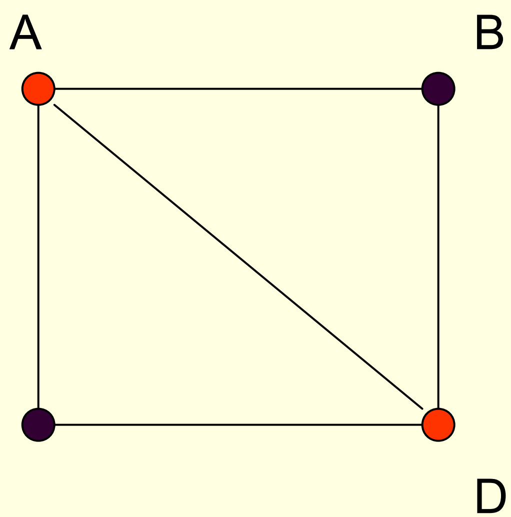

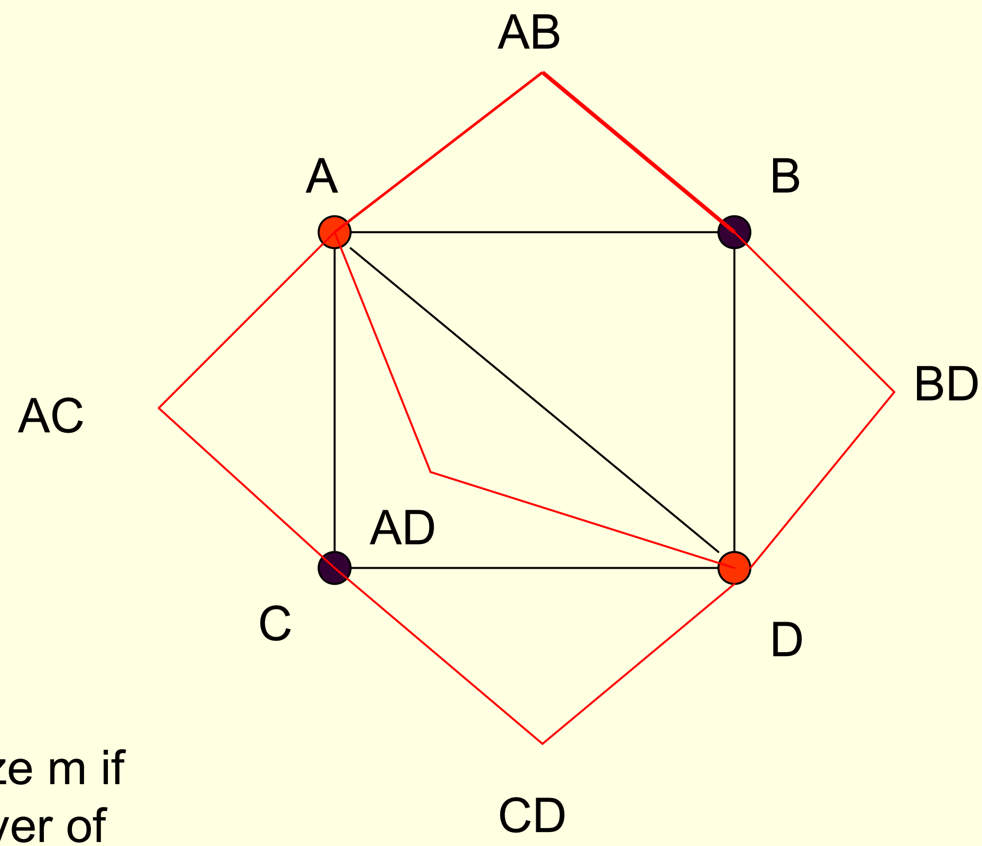

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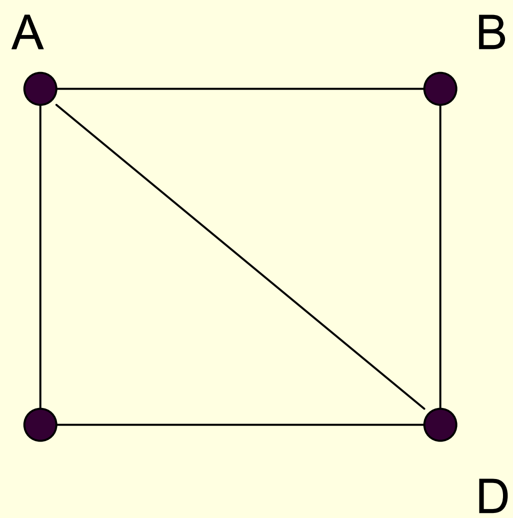

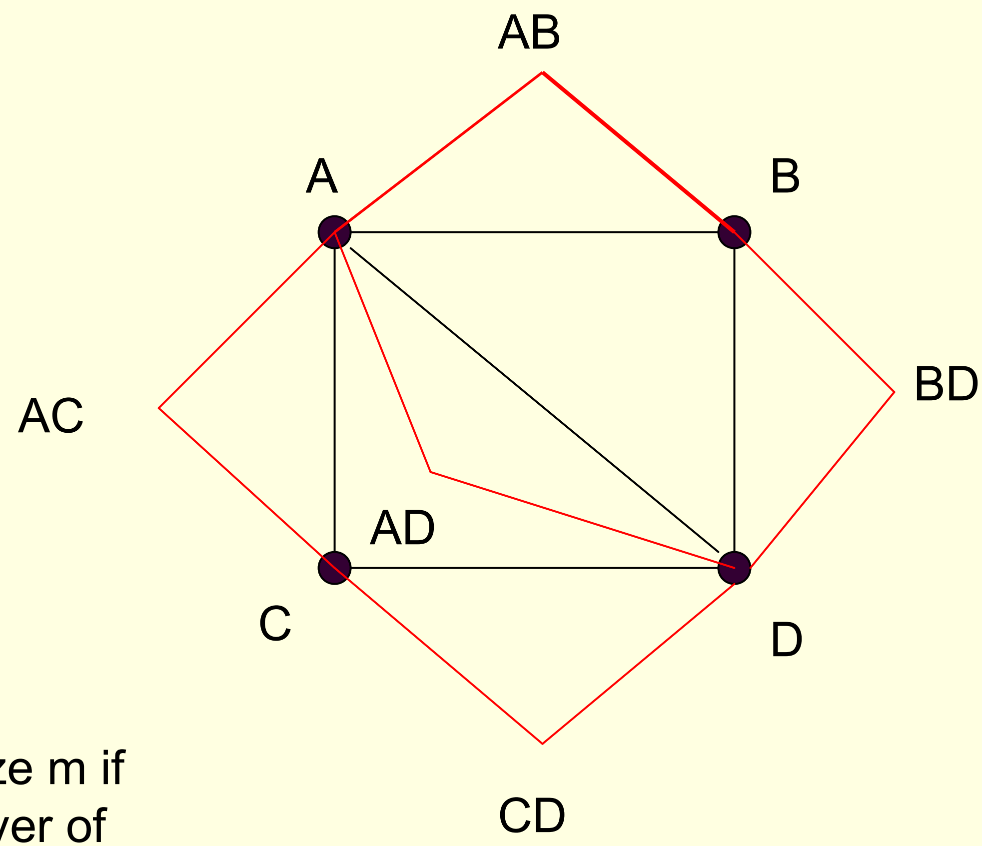

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NP Complete Problems

- Hamiltonian Circuit Problem
  - Determine whether a given graph has a Hamiltonian circuit.
- Travelling Salesman
  - Find the shortest tour through n cities with known positive integer distances between them.
- Knapsack Problem
  - Find the most valuable subset of n items of given positive integer weights and values that fit into a knapsack of a given positive integer capacity.
- Partition Problem
  - Given n positive integers, determine whether it is possible to partition them into two disjoint subset with the same sum.
- Bin Packing
  - Given n items whose sizes are positive rational numbers not larger than 1, put them into the smallest number of bins of size 1.
- Graph Coloring
  - For a given graph, find its chromatic number (the smallest number of colors that need to be assigned to the graph's vertices so that no two adjacent vertices are assigned the same color.
- Integer Linear Programming
  - Find the maximum (or minimum) value of a linear function of several integer-valued variables subject to a finite set of constrains in the form of linear equalities and/or inequalities.

Dealing with NP Complete problems

- Proving that a given problem is NP-Complete does not make the probem go away!! Udi Manber

- An NP-Complete problem cannot be solved precisely in polynomial time
  - We make compromises in terms of optimality, robustness, efficiency, or completeness of the solution.

- Approximation algorithms do not lead to optimal solutions
  - Probabilistic algorithms
  - Branch and bound
  - Backtracking

Backtracking

- The Hamiltonian Circuit Problem
- The Subset problem

Branch and Bound

- Job Assignment
- Knapsack