Flow Networks

Topics

Flow Networks
Residual networks
Ford-Fulkerson's algorithm
Ford-Fulkerson's Max-flow Min-cut Algorithm

Chapter 7

Algorithm Design ***Kleinberg and Tardos***

Flow Networks

A directed graph can be interpreted as a flow network to analyse material flows through networks.

Material courses through a system from a source (where it is produced) to a sink (where it is consumed).
Examples :

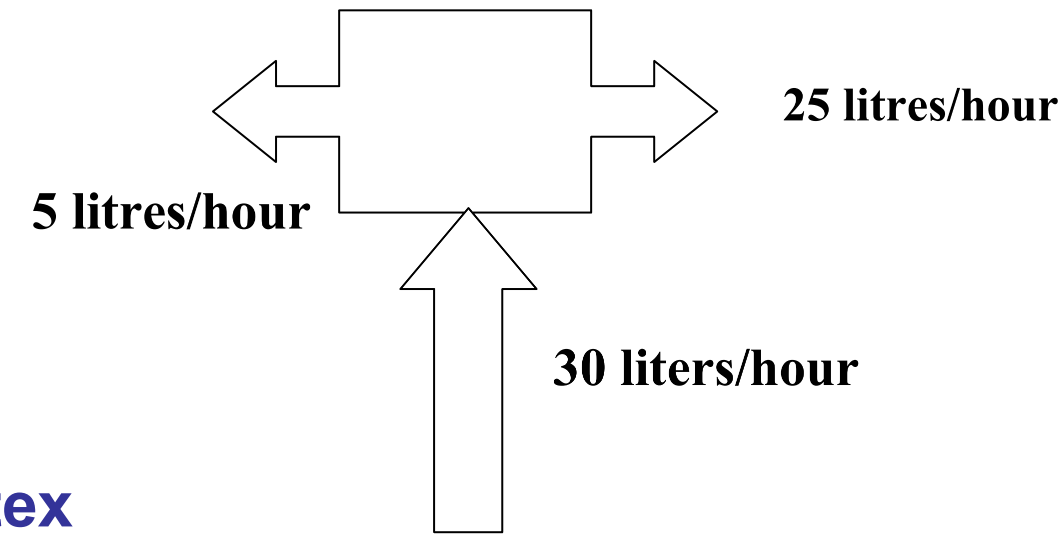
- **Water through pipelines**
- **Newspapers through distribution system**
- **Electricity through cables**
- **Cars on a production line**
  - **on roads**

The source produces the material at a steady rate .
The sink consumes the material at a steady rate

Flow: the rate at which the material moves from one point to another

100 litres of water per hour in a pipe

30 Amperes of electric current in a circuit

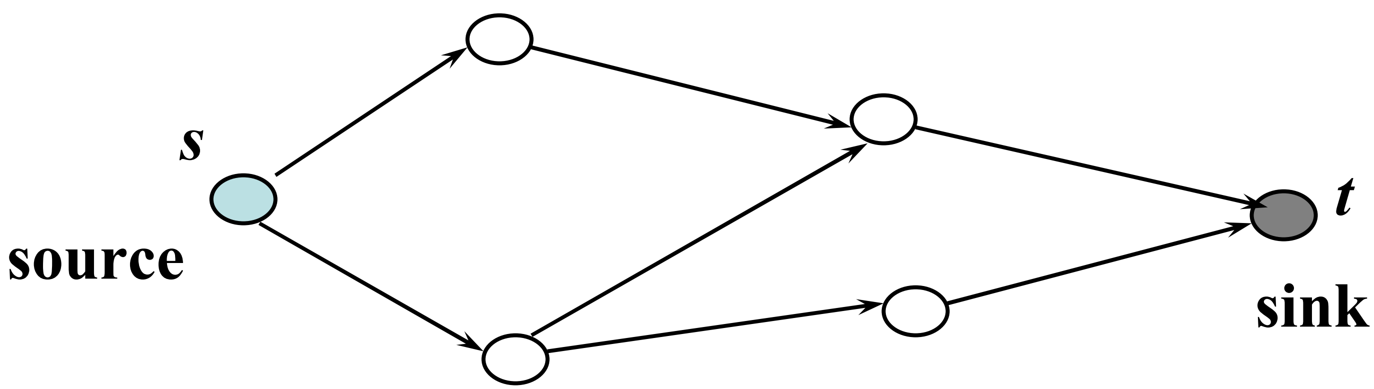


The rate at which a material enters a vertex = the rate at which the material leaves the vertex

The flow network *G =(V,E)* is a directed graph in which each edge *(u,v)* $\in$ *E* has a nonnegative capacity c(u,v) $\geq$ 0. If *(u,v)* $\notin$ *E* then *c(u,v)* = 0.

A flow network has a source vertex *s*, and a sink vertex *t*.

For every vertex *v* $\in$ *V* there is a path from *s* to *v* and *v* to *t* in a connected graph.



A flow in G is a real-valued function $f : V \times V \rightarrow R$ that satisfies the following three properties:

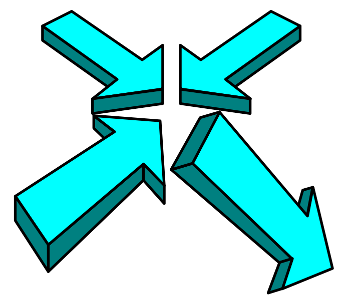
1. Capacity constraint : For all $u,v \in V$, we require $f(u,v) \leq c(u,v)$.
The net flow from one vertex to another must not exceed the given capacity.

2. Skew symmetry : For all $u,v \in V$, we require $f(u,v) = -f(v,u)$.

The net flow from a vertex u to a vertex v is the negative of the net flow in the reverse direction.
The net flow from a vertex to itself is zero for all $u \in V$, that is $f(u,u) = 0$.

3. Flow conservation : For all $u \in V - \{s,t\}$, we require

$$\sum_{v \in V} f(u,v) = 0$$



The total net flow out of a vertex other than the source or sink is zero.

The quantity *f(u,v)* can be negative or positive, it is called the net flow from vertex *u* to *v*.

The value of a flow is defined as

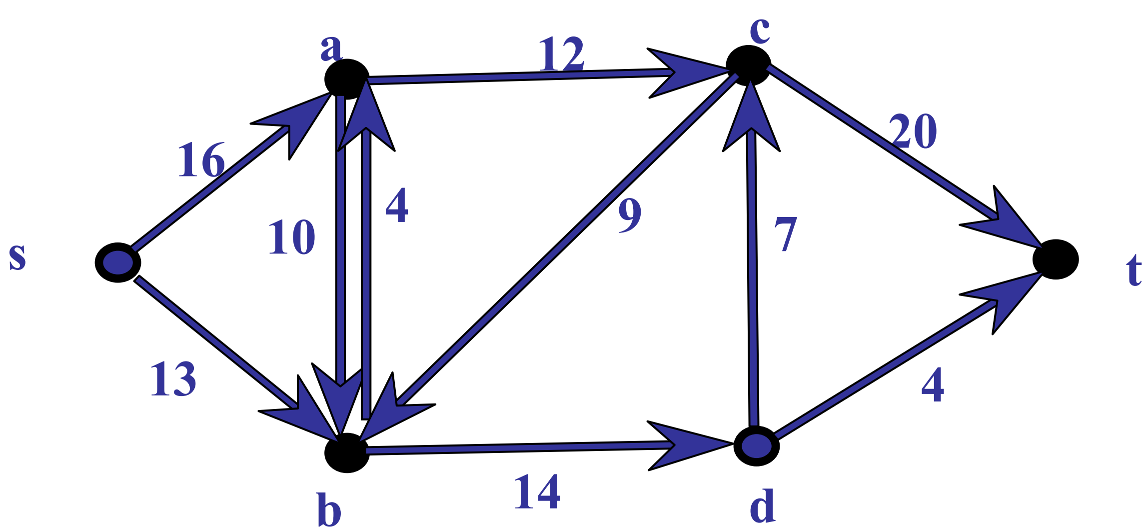
$$\left|f\right| = \sum_{v \in V} f(s,v)$$

In the maximum-flow problem, we are given a flow network G with source *s* and sink *t*, and we wish to find a flow of maximum value from *s* to *t*.

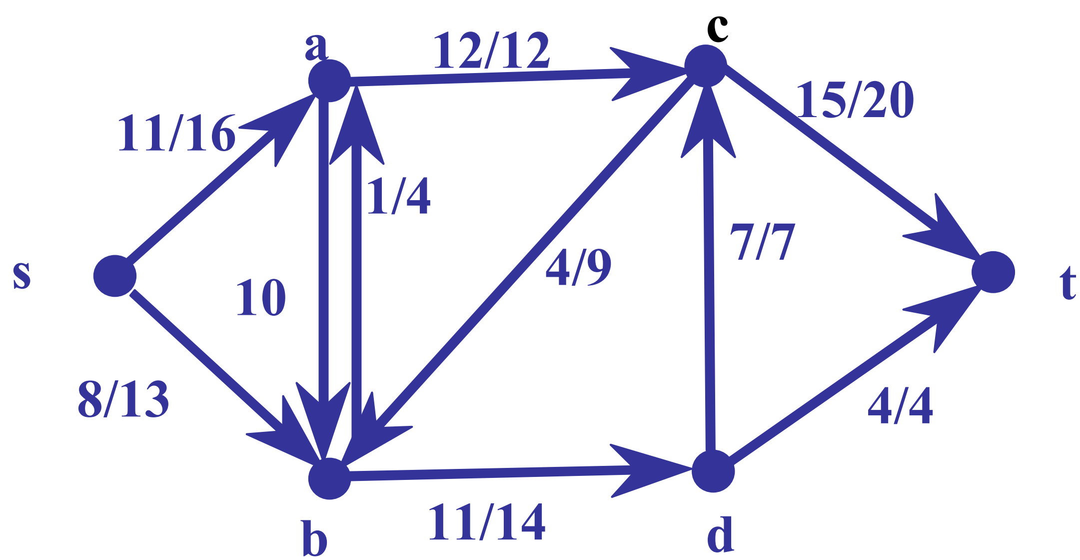
There is no net flow between *u* and *v* if there is no edge between them.
If *(u,v)* $\notin$ *E* and *(v,u)* $\notin$ *E*, then *c(u,v)* = *c(v,u)* = 0.
Hence, the capacity constraint, *f(u,v)* $\leq$ 0 and *f(v,u)* $\leq$ 0.
By skew symmetry, *f(u,v)* = -*f(v,u)*,
therefore, *f(u,v)* + *f(v,u)* = 0.

Nonzero net flow from vertex *u* to vertex *v* implies that *(u,v)*$\in$*E* or *(v,u)*$\in$*E* (or both).

Consider the network *G=(V,E)* shown in the figure below. The network is for a transport system that transports crates of an item from source vertex *s* to sink vertex *t* through a number of intermediate points. Each edge *(u,v)* $\in$ *E* in the network is labeled with its capacity *c(u,v).*



Let us consider a flow in *G*, $|f|$=19
If *f(u,v)* >0, edge (*u,v*) is labeled *f(u,v)/c(u,v)*
If *f(u,v)* $\leq$ 0, the edge is labeled by its capacity only.



The positive net flow entering a vertex *v* is defined by

$$\sum_{\substack{u \in V \\ f(u,v)>0}} f(u,v)$$

Initially, c (a ,b) = 8, and c (b, a) = 3 -- Fig. a.
f (a, b) = 5 and f (b, a) = 2, -- Fig. b
the net flow is shown as 3/8 in direction a to b – Fig. c

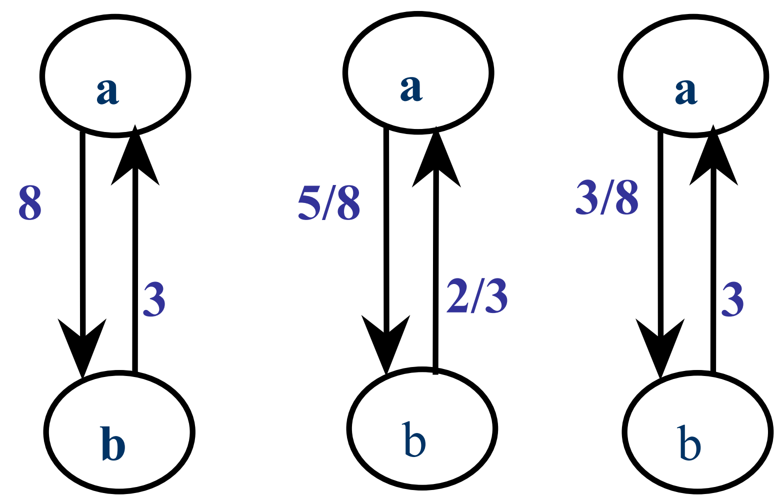


Fig.a Fig.b Fig.c

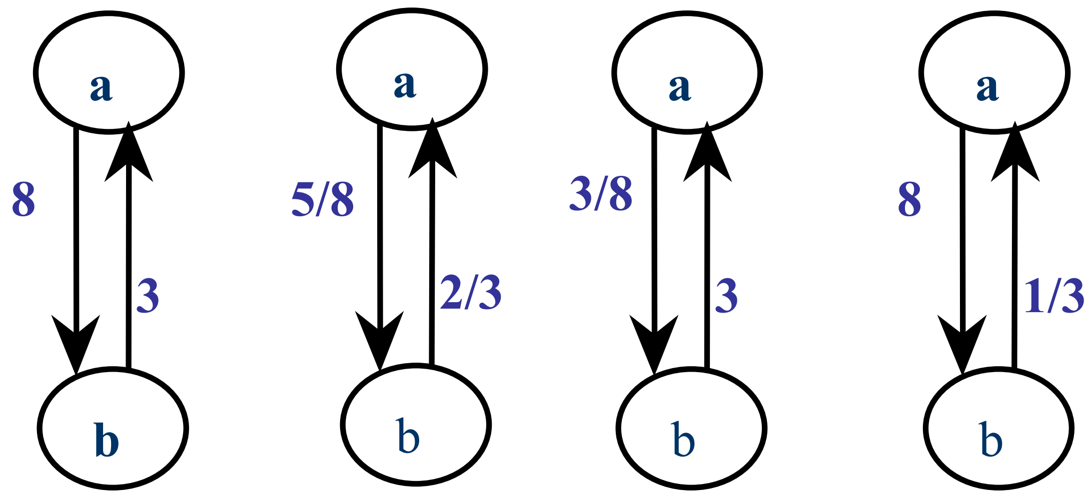


Fig.a Fig.b Fig.c Fig.d

If we increase the flow from b to a from 2 to 6 then the net flow is 1/3 in the direction b to a as shown in Fig. d.

The Ford Fulkerson method

The method is iterative,
Starts with *f(u,v)* for *(u,v)* $\in$ *V*, initial flow of value 0.
The method is based on the augmenting path which is defined as a path from *s* to *t* along which we can push more flow and then augment flow along this path.

Procedure Ford_Fulkerson_method(*G,s,t*)

1. ***f*** $\leftarrow$ **0;**
2. **while there exists an augmenting path p**
3. **do augment flow along path p**
4. **return f**

Residual Networks

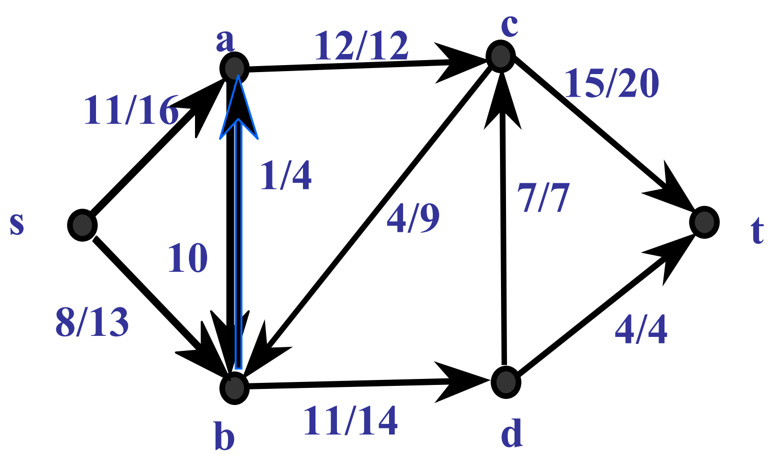
Consider a flow network *G(V,E)* with source *s* and sink *t* and let *f* be a flow in *G*.
Consider a pair of vertices $u,v \in V$.
Residual capacity between u and v is given by

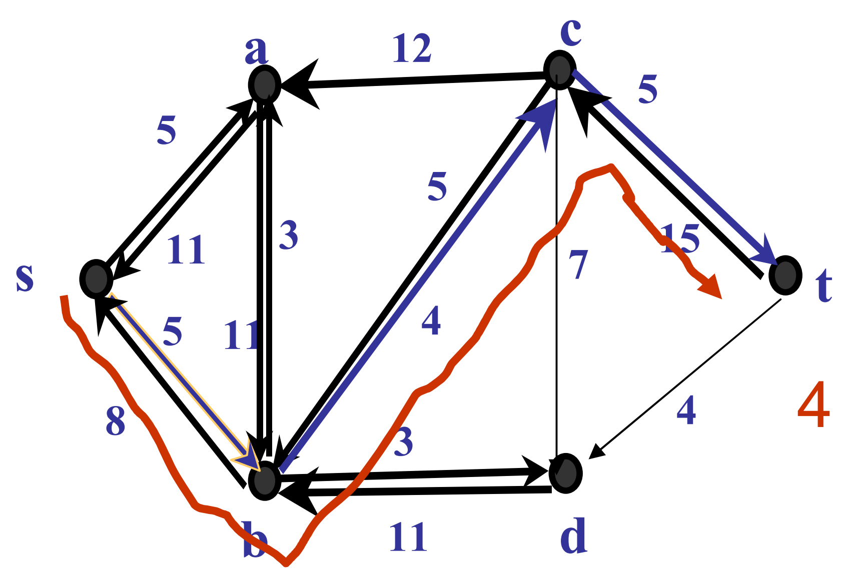
$$r(u,v) = c(u,v) - f(u,v)$$

- **the additional net flow we can push from *u* to *v* before exceeding the capacity.**
  For example, if $c(u,v) = 25$ and $f(u,v) = 19$, then $r(u,v) = 6$.

If $f(u,v) < 0$ then $r(u,v) > c(u,v)$

Given a flow network *G=(V,E)* and a flow *f*, the residual network of *G* induced by *f* is $G_f=(V,E_f)$,
where $E_f = \{(u,v) \in V \times V : r(u,v) > 0\}$





Each edge in the residual network can admit positive net flow only.

The residual network may include several edges that are not in the original network, $(u,v) \in E_f$ and $(u,v) \notin E$ is possible (E_f is not a subset of E). However, (u,v) appears in G_f only if $(v,u) \in E$ and there is a positive flow from v to u. Because the net flow $f(u,v)$ is negative,

$$r(u,v) = c(u,v)-f(u,v) > 0 \text{ and } (u,v) \in E_f$$

An edge (*u*,*v*) can appear in a residual network only if at least one of (*u*,*v*) and (*v*,*u*) appears in the original network.

$$|E_f| \leq 2|E|$$

Augmenting Paths

It is a simple path from *s* to *t* in G_f. Each edge (*u*,*v*) on an augmenting path admits some additional positive net flow from *u* to *v* without violating the capacity constraint on the edge. The residual capacity of a path *p* is given by,

$$r(p) = min \{ r(u,v) : (u,v) \text{ is in } p \}$$

Let's define a flow function f_p,

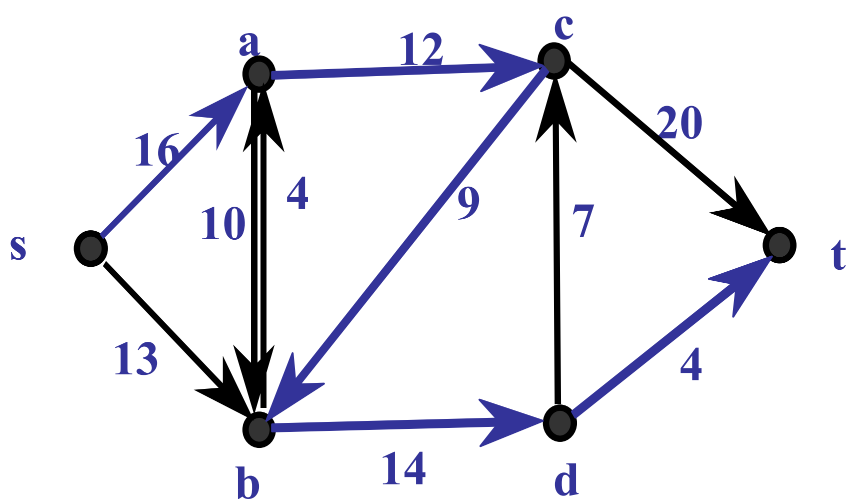
$$f_p = \begin{cases} r(p) \ \ if \ \ (u,v) \ \ is \ \ on \ \ p, \\ -r(p) \ \ if \ \ (v,u) \ \ is \ \ on \ \ p \\ 0 \ \ otherwise \end{cases}$$

f_p is a flow in G_f with value $|f_p|$ = *r(p)* >0.
If we add f_p to *f*, we get another flow in *G* whose value is closer to the maximum.

Algorithm

Procedure Ford-Fulkerson(*G,s,t*)
Input : Flow Network *G(V,E)*
Output : Maximum flow for the given network

```
1.for each edge (u,v) ∈ E
2.    do    f[u,v] ← 0;
3.          f[v,u] ← 0;
4.while there exists a path p from s to t in the
                          residual network G_f
5.    do    r(p) ← min {r(u,v) : (u,v) is in p};
6.          for each edge (u,v) in p
7.               do    f[v,u] ← - f[u,v];
8.                     f[u,v] ← f[u,v] + r(p);
9.return
```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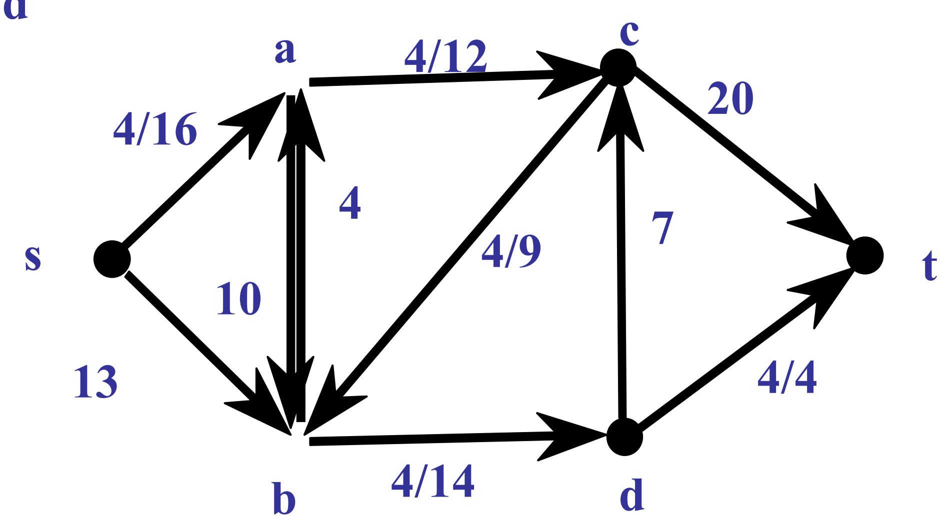
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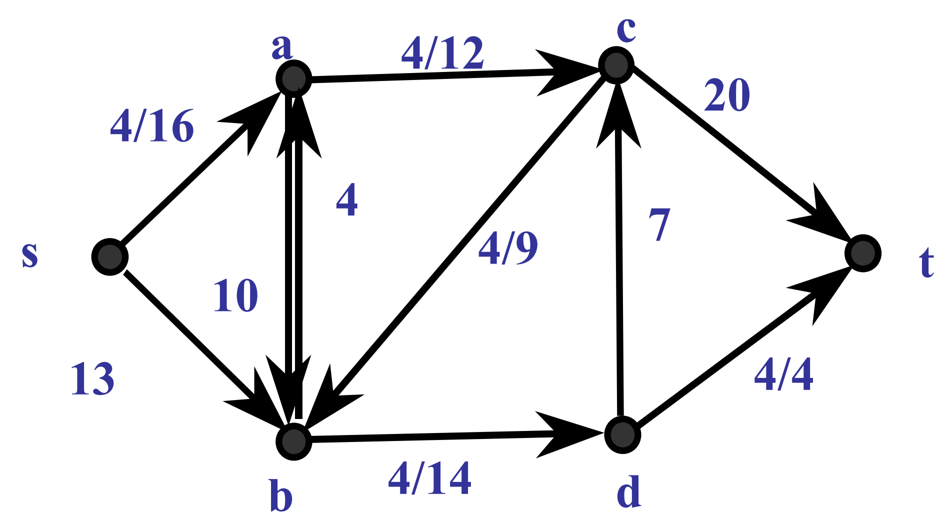
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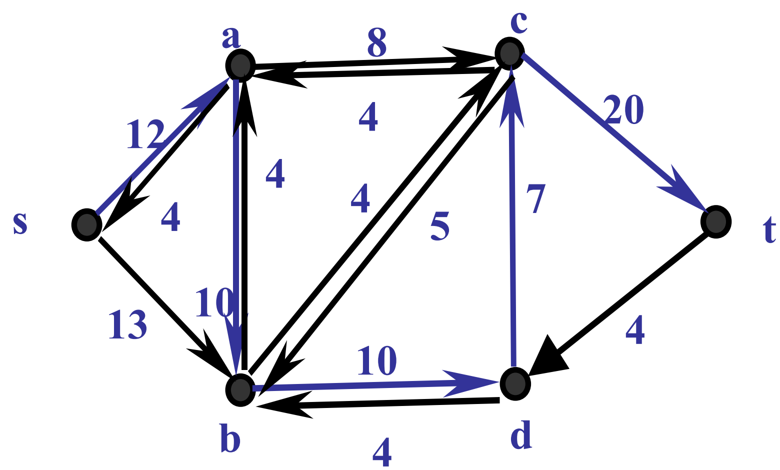
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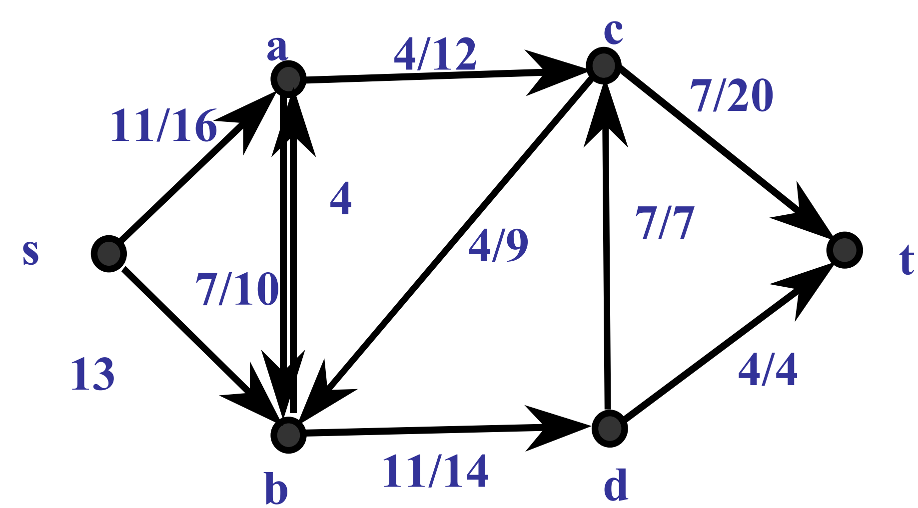
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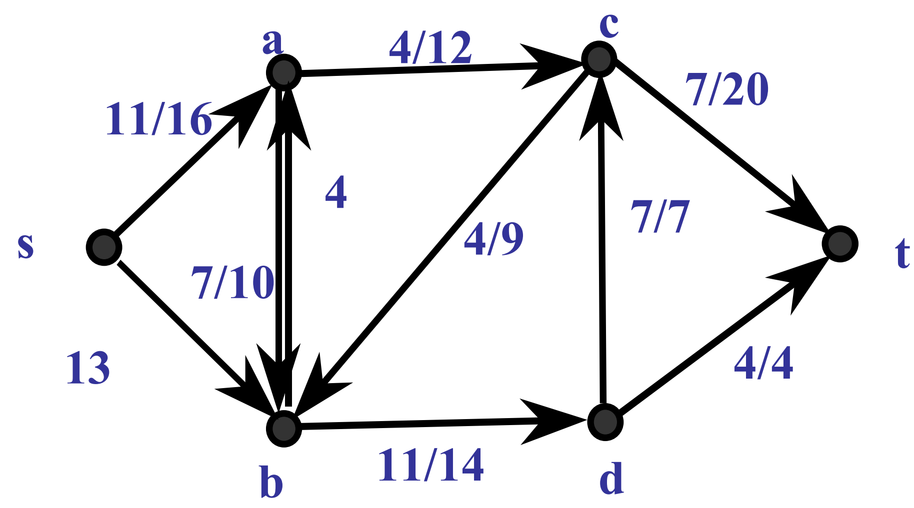
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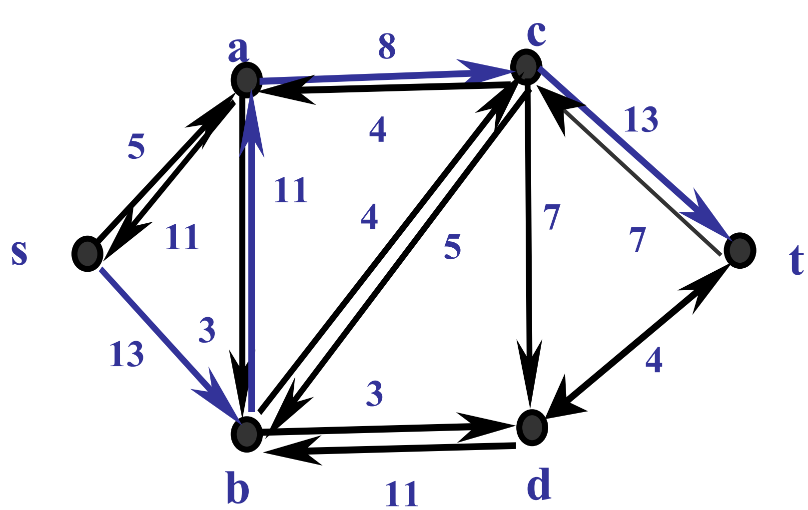
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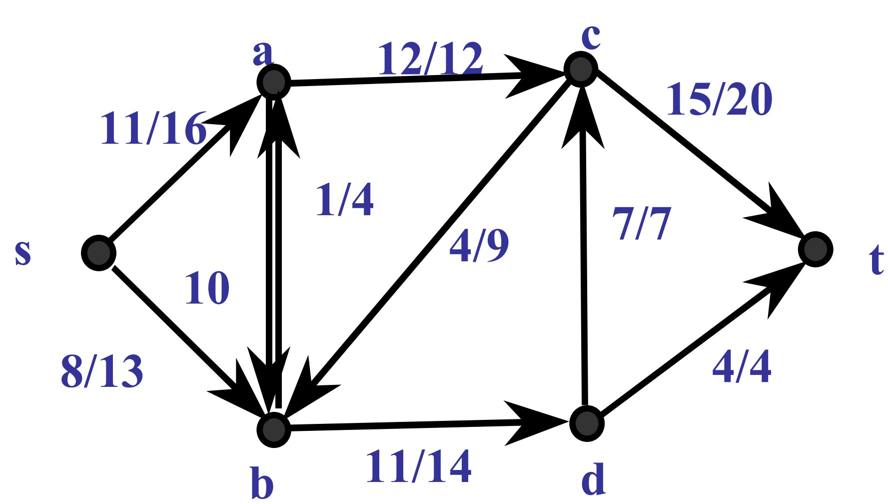
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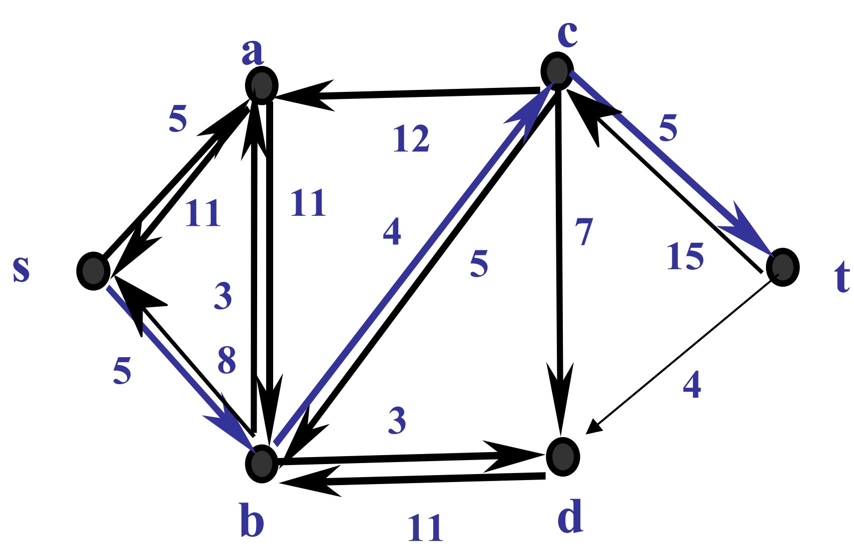
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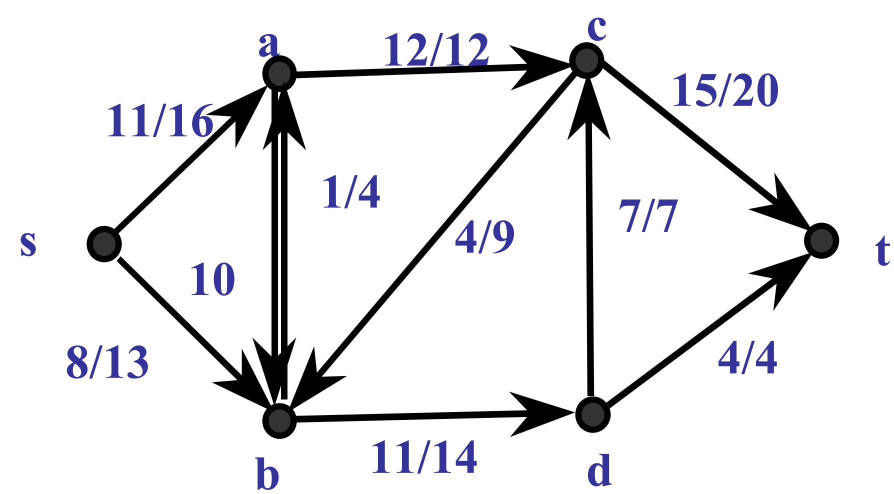
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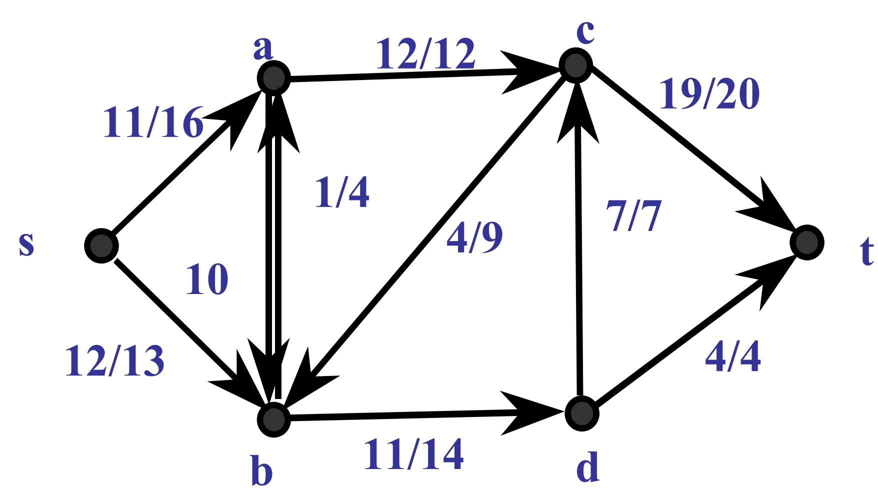
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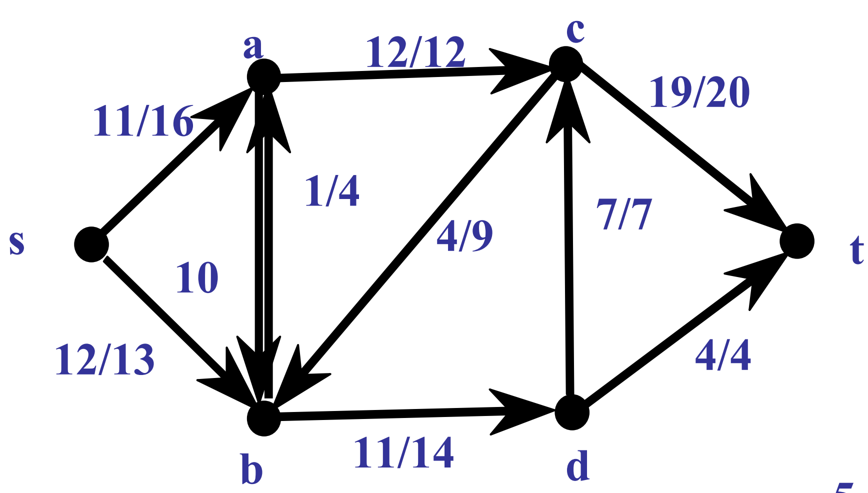
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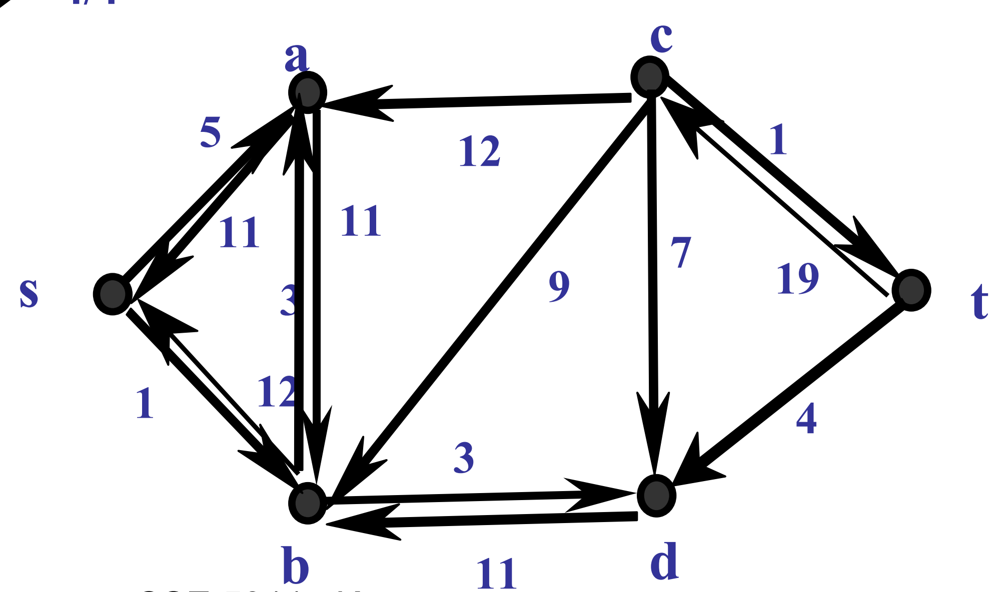
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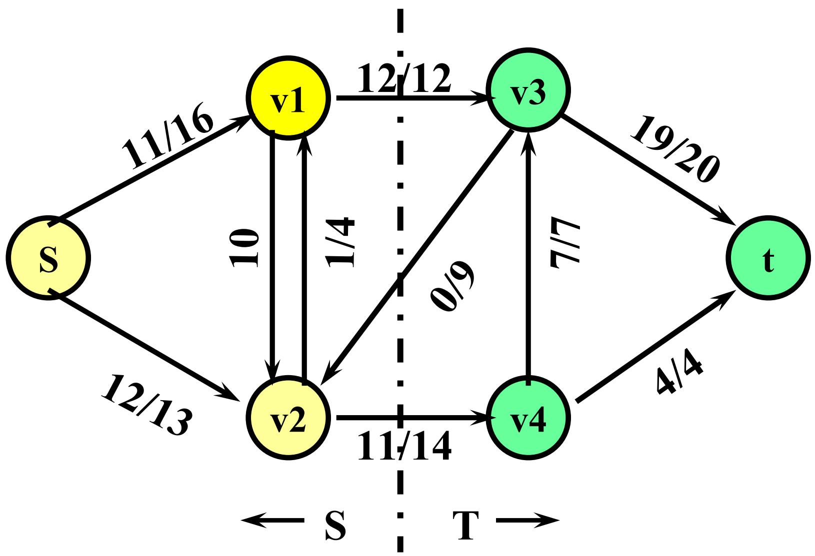
Ford Fulkerson – cuts of flow networks

New notion: cut (S,T) of a flow network

A cut (S,T) of a flow network $G=(V,E)$ is a partition of V in to S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.



In the example:

$S = \{s, v1, v2\}$, $T = \{v3, v4, t\}$

Net flow $f(S,T) = f(v1,v3) + f(v2,v4) + f(v2,v3)$

$= 12 + 11 + (-0) = 23$

Capacity $c(S,T) = c(v1,v3) + c(v2,v4)$

$= 12 + 14 = 26$

Implicit summation notation: $f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v)$

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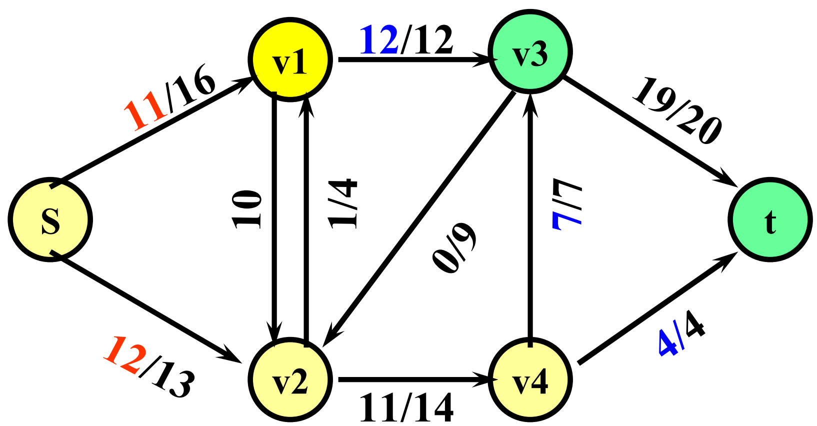
Ford Fulkerson – cuts of flow networks

Lemma:

The value of a flow in a network is the net flow across any cut of the network

$$f(S,T) = |f|$$



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Ford Fulkerson – cuts of flow networks

Assumption:

The value of any flow f in a flow network *G* is bounded from above by the capacity of any cut of *G*

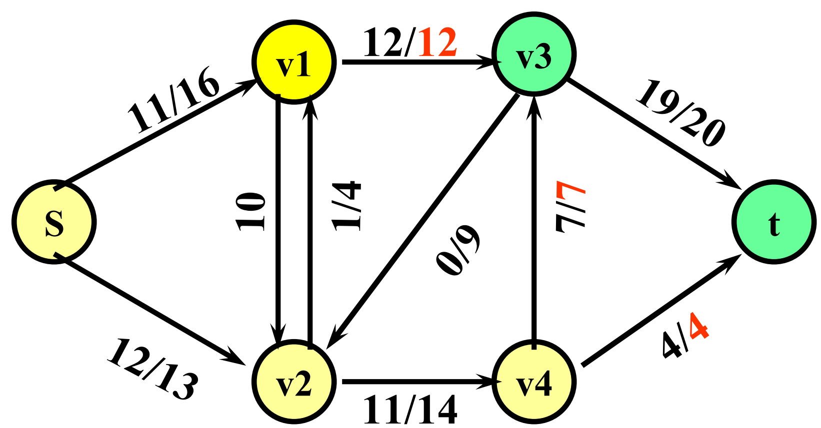
Lemma: $|f| \leq c\,(S, T)$

$$|f| = f\,(S, T)$$

$$= \sum_{u \in S} \sum_{v \in T} f\,(u, v)$$

$$\leq \sum_{u \in S} \sum_{v \in T} c\,(u, v)$$

$$= c\,(S, T)$$



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F. Fulkerson: Max-flow min-cut theorem

If f is a flow in a flow network G = (V,E) with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some *cut* (S, T) of G.

proof:

(1) $\Rightarrow$ (2):

We assume for the sake of contradiction that f is a maximum flow in G but that there still exists an augmenting path p in G_f.

Then as we know from above, we can augment the flow in G according to the formula: $f' = f + f_p$. That would create a flow f' that is strictly greater than the former flow f which is in contradiction to our assumption that f is a maximum flow.

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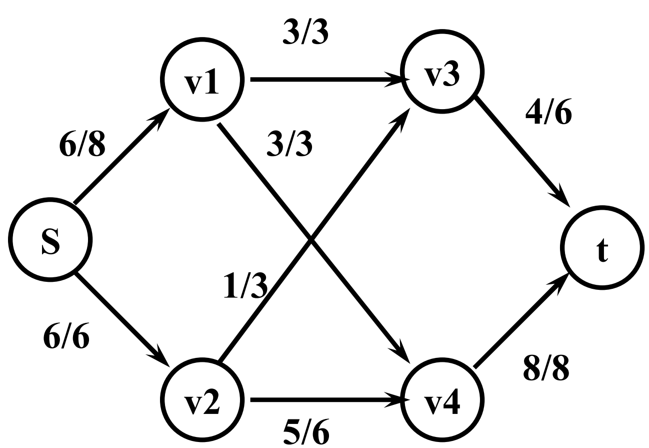
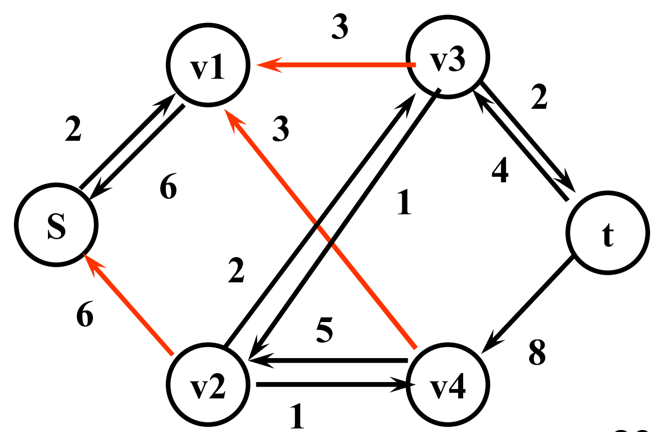
proof:

(2) $\Rightarrow$ (3):

original flow network G



residual network G_f



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proof:

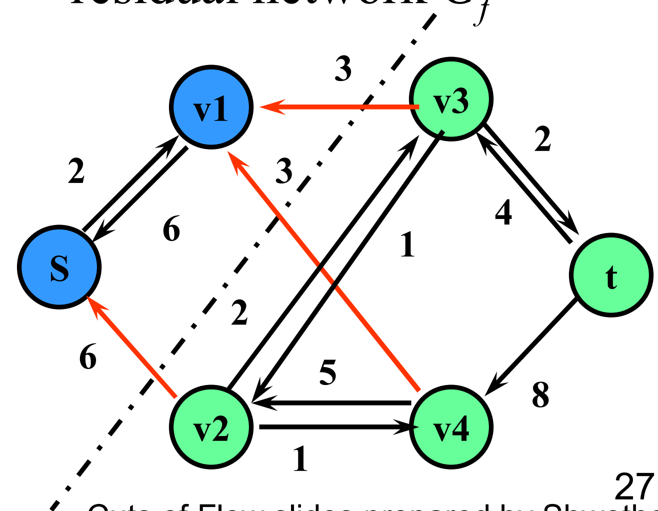
(2) $\Rightarrow$ (3):

Define

$\boldsymbol{S} = \{v \in V \mid \exists$ **path** p **from** s **to** v **in** $G_f\}$

$\boldsymbol{T} = V \setminus S$ (note $t \notin S$ according to (2))

residual network G_f



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1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. | f | = c (S, T) for some cut (S, T) of G.

proof:

(2) $\Rightarrow$ (3):

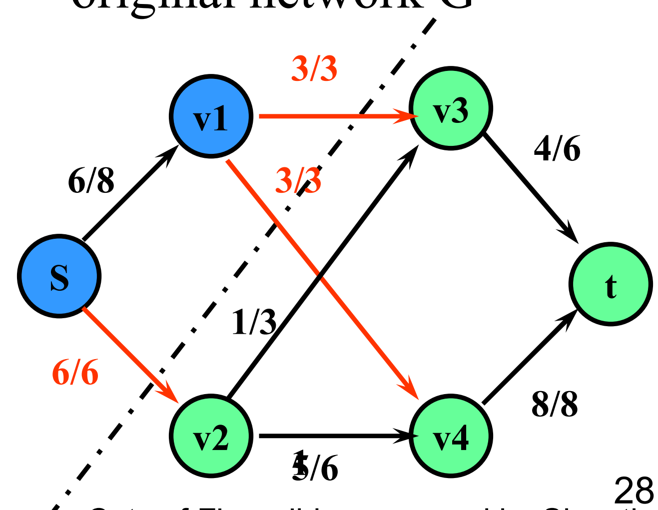
$S = \{v \in V \mid \exists$ path p from s to v in $G_f\}$

$T = V \setminus S$ (note $t \notin S$ according to (2))

$\Rightarrow$ for $\forall\, u \in S, v \in T$: f (u, v) = c (u, v) (otherwise $(u, v) \in E_f$ and $v \in S$)

$\Rightarrow$ | f | = f (S, T) $\leq$ c (S, T)

original network G



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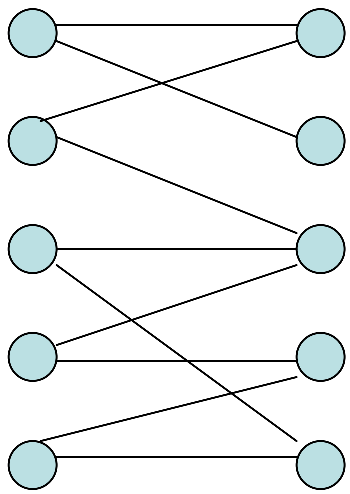
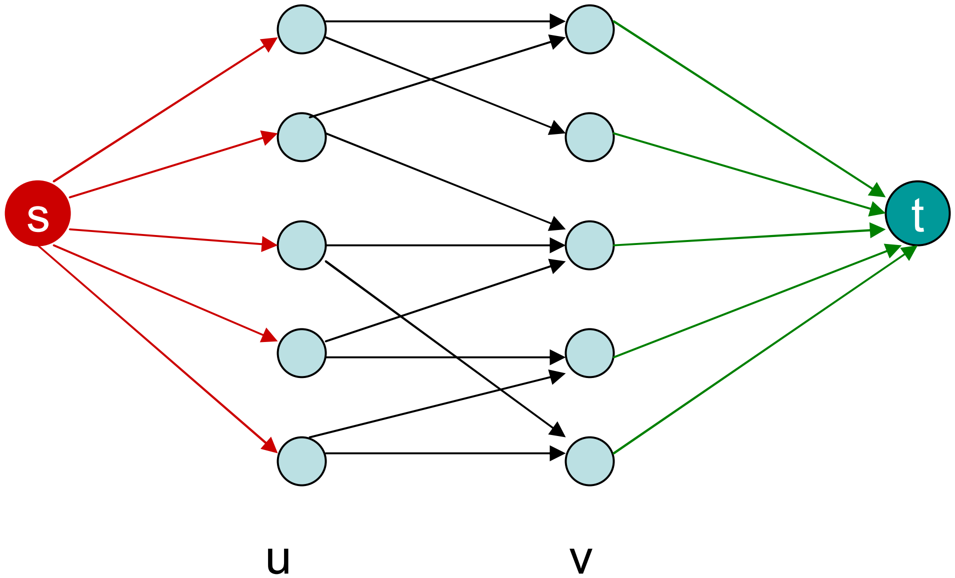
- Suppose that each source s_i in a multisource, multisink problem produces exactly p_i units of flow, so that $f(s_i, V) = p_i$. Suppose that each sink t_j consumes exactly q_j units so that $f(V, t_j) = q_j$, where . Show how to convert the problem of finding a flow f that obeys these additional constraints into the problem of finding a maximum flow in a single-source, single-sink flow network.
- Given a flow network $G = (V, E)$, let $f1$ and $f2$ be functions from $V \times V$ to **R**. The flow sum $f1 + f2$ is the function from $V \times V$ to **R** defined by $(f1 + f2)(u, v) = f1(u, v) + f2(u, v)$ for all $u, v \in V$. If $f1$ and $f2$ are flows in G, which of the three flow properties must the flow $f1 + f2$ satisfy, and which might it violate?
- The edge connectivity of an undirected graph is the minimum number k of edges that muct be removed to disconnect the graph. For example, the edge connectivity of a tree is 1, and the edge connectivity of a cyclic chain of vertices is 2. Show that how the edge connectivity of an undirected graph $G = (V,E)$ can be determined by running a maximum-flow algorithm on at most $|V|$ flow networks, each having $O(V)$ vertices and $O(E)$ edges.

Bipartite Matching

- Finding a matching M in *G* of largest size
- A bipartite graph $G = (V,E)$ is an undirected graph whose node set is partitioned into two sets X and Y such that $V = X \cup Y$. Every edge $e \in E$ has one end in X and the other end in Y.
- A matching M in G is a subset of the edges $M \subseteq E$ such that each node $v \in V$ appears in at most one edge in M.

Bipartite graph and Flow Network





Each edge has a capacity of ONE

F. Fulkerson: Max-flow min-cut theorem

If f is a fow in a flow network $G = (V,E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c\,(S, T)$ for some cut (S, T) of G.

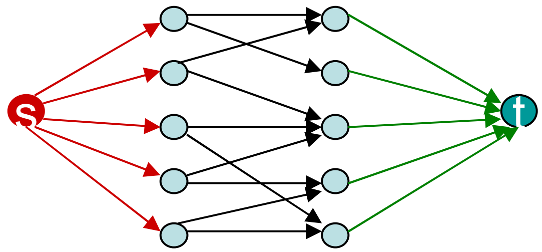
proof:

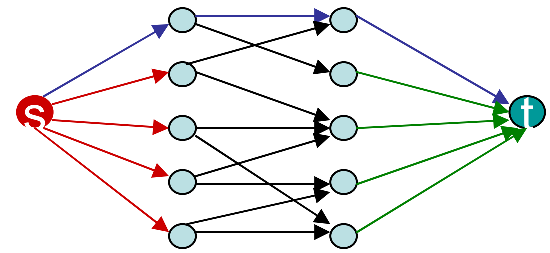
(3) $\Rightarrow$ (1):

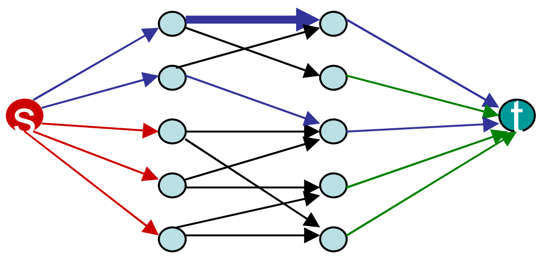
$|f| = f\,(S, T) \le c\,(S, T)$

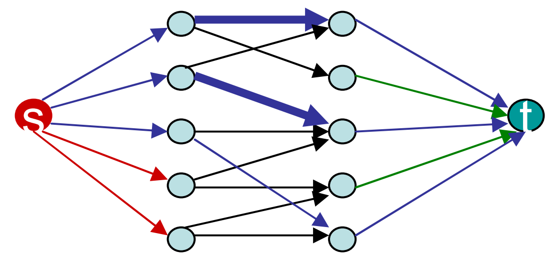
the statement of (3) : $|f| = c\,(S, T)$ implies that f is a maximum flow

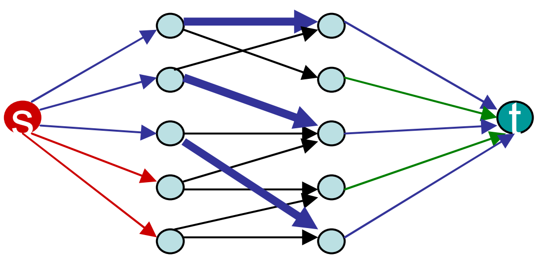
Cuts of Flow slides prepared by Shwetha and Pradeep

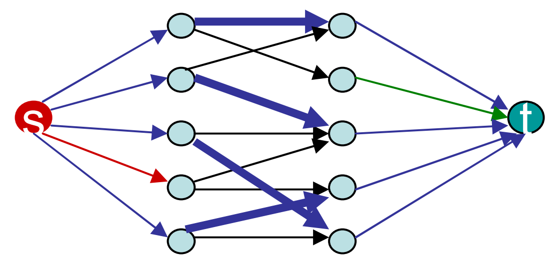












v1-u1

v2-u3

v3-u5

v5-u4