Computational Geometry

TOPICS
- Preliminaries
- Point in a Polygon
- Polygon Construction
- Convex Hulls

Further Reading

Geometric Algorithms

Geometric Algorithms find applications in such areas as

- **Computer Graphics**
- **Computer Aided Design**
- **VLSI Design**
- **GIS**
- **Robotics**

We will study algorithms dealing with

points, lines, line segments, and polygons

In particular, the algorithms will

- ***Determine whether a point is inside a Polygon***
- ***Construct a Polygon***
- ***Determine Convex Hulls***

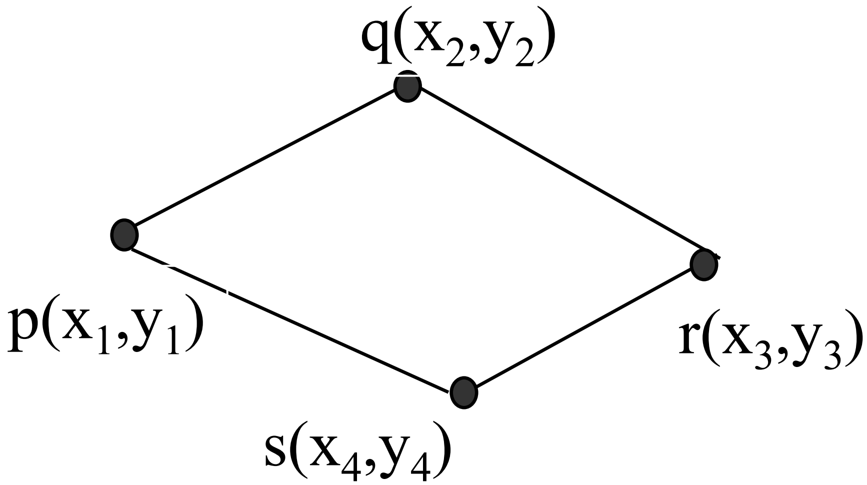
Preliminaries:

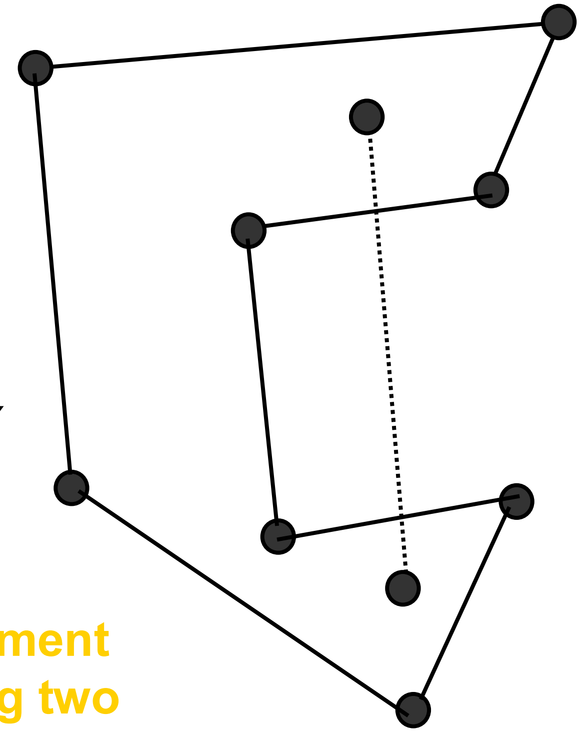
A point p is represented as a pair of coordinates (x,y)
A line is represented by a pair of points
A path is a sequence of points $p_1, p_2, \ldots p_n$ and the line segments connecting them,

$$p_1\text{-}p_2,\ p_2\text{-}p_3,\ \ldots,\ p_{k-1}\text{-}p_k.$$

A closed path whose last point is the same as the first is a polygon.
A simple polygon is one whose corresponding path does not intersect itself. It encloses a region in the plane.

A convex Polygon is a polygon such that any line segment connecting two points inside the polygon is itself entirely in the polygon.
The convex hull of a set of points is defined as the smallest convex polygon enclosing all the given points.

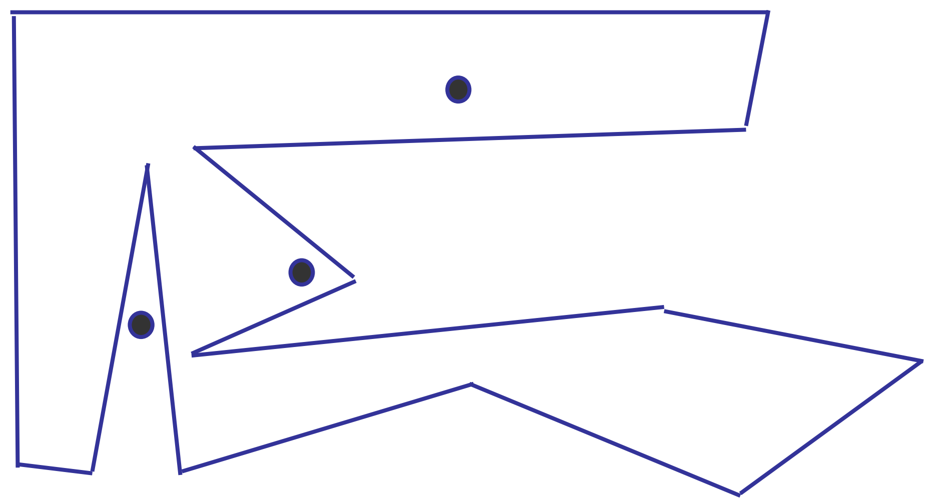




This is not a convex polygon

A line segment connecting two points:
The points are inside the polygon
The line segment is not entirely in the polygon

Determining whether a *point* is inside a polygon



Given a simple polygon polygon P, and a point q, determine whether the point is inside or outside the polygon. (a non-convex polygon)

Procedure **Point_in_a_Polygon(P,q)**

Input : P (a simple polygon with vertices p_1, p_2, p_3, and edges e_1, e_2, e_3, … e_n and q (x_0, y_0) a point.

Output: INSIDE (a Boolean variable, True if q is inside P, and false otherwise)

Count $\leftarrow$ 0;

for all edges e_i of the polygon do

if the line $x = x_0$ intersects e_i then

$y_i \leftarrow$ y coordinate of the intersection between lines e_i and $x = x_0$;

if $y_i > y_0$ then

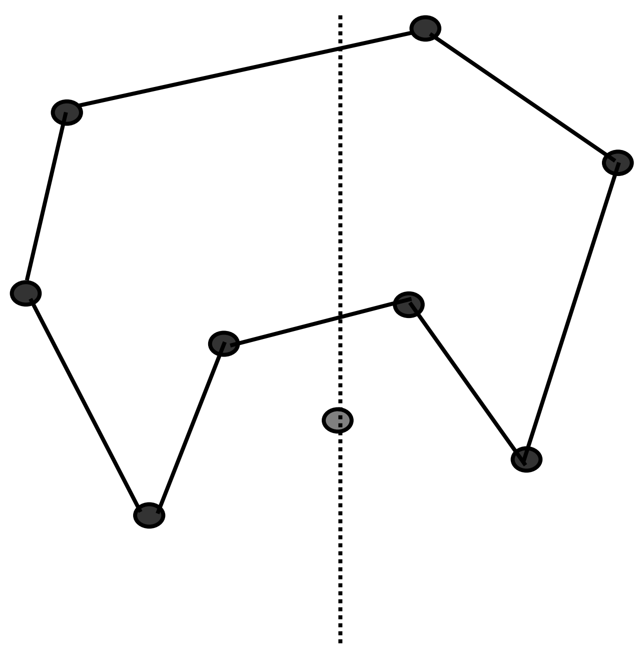
Count $\leftarrow$ Count +1;

if count is odd then INSIDE $\leftarrow$ TRUE;

else INSIDE $\leftarrow$ FALSE

This does not work if the line passes through terminal points of edges

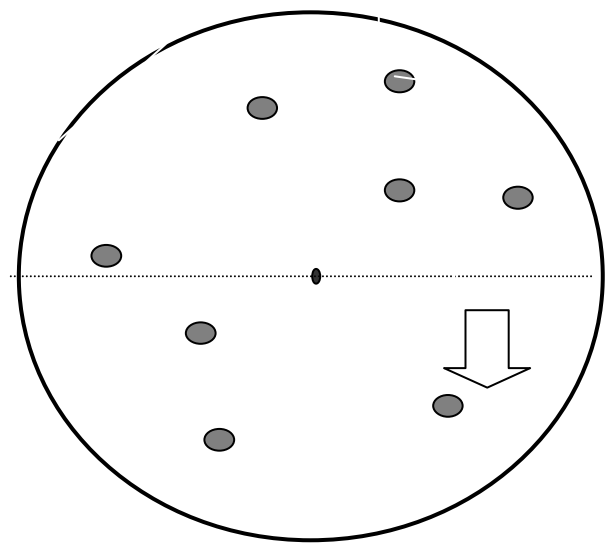
It takes constant time to perform an intersection between two line segments.
The algorithm computes n such intersections, where n is the size on the polygon.
Total running time of the algorithm, O(n).



Constructing a Simple Polygon

Given a set of points in the plane, connect them in a simple closed path.

> **Consider a large circle that contains all the points.**
> **Scan the area of C by a rotating line.**
> **Connect the points in the order they are encountered in the scan.**



Procedure Simple_Polygon

Input : $p_1, p_2, \ldots p_n$ (points in the polygon)
Output : P (a simple polygon whose vertices $p_1, p_2, \ldots p_n$ are in some order)

p1 $\leftarrow$ the point with the max 'x' value.

1. **for** i $\leftarrow$ 2 **to** n
2. $\alpha_i \leftarrow$ angle between line p_1-p_i and the x-axis;
3. sort the points according to the angles (use the corresponding priority for the point and do a heapsort)
4. P is the polygon defined by the list of points in the sorted order.

Complexity : Complexity of the sorting algorithm.

Convex Hulls

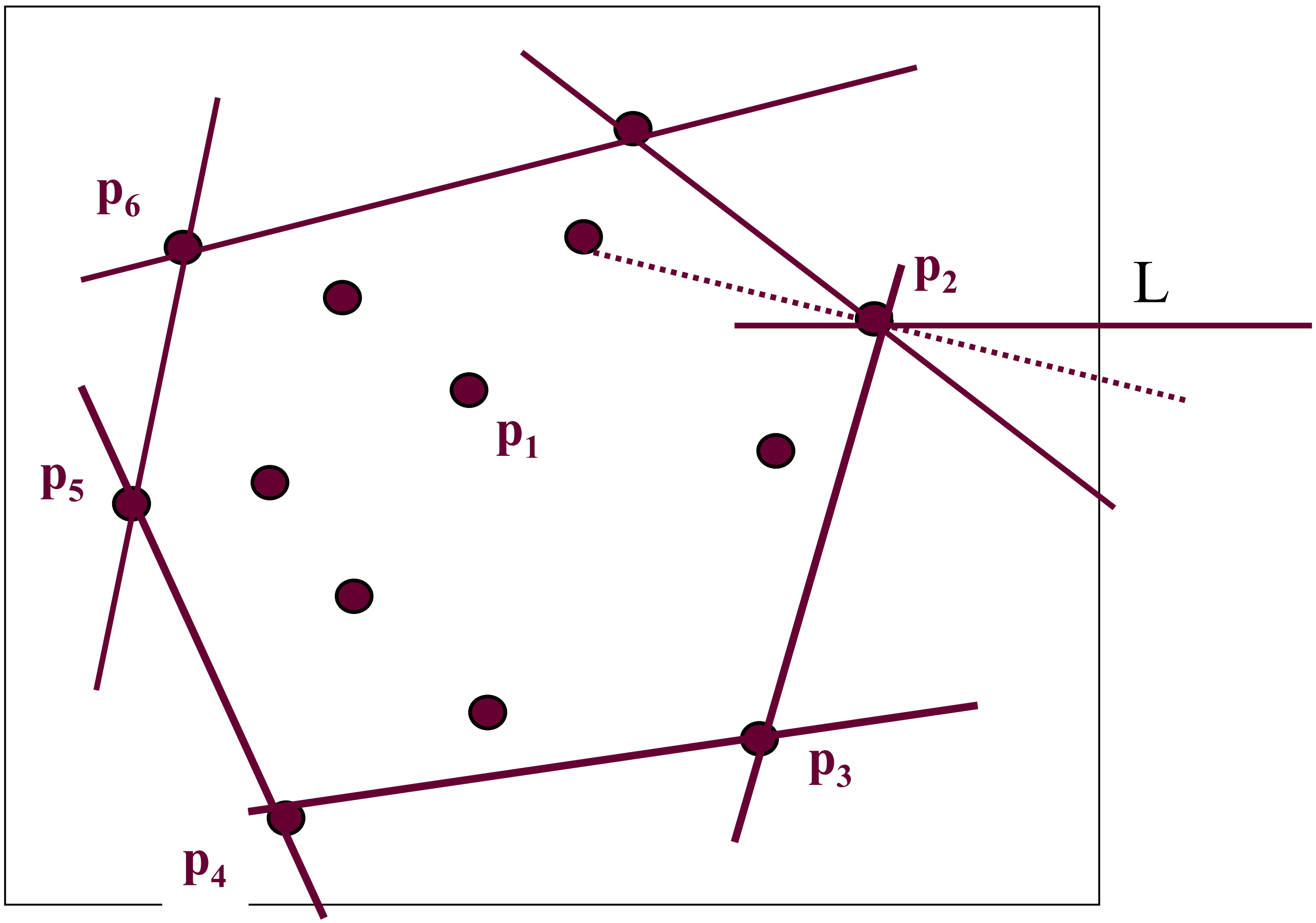
The convex hull of a set of points is defined as the smallest convex polygon enclosing all the points in the set.

The convex hull is the smallest region encompassing a set of points.
A convex hull can contain as little as three and as many as all the points as vertices.

Problem Statement : Compute the convex hull of n given points in the plane.

There are two algorithms

Gift Wrapping $O(n^2)$
Graham's Scan $O(n \log n)$


p_6
p_2
L
p_1
p_5
p_3
p_4

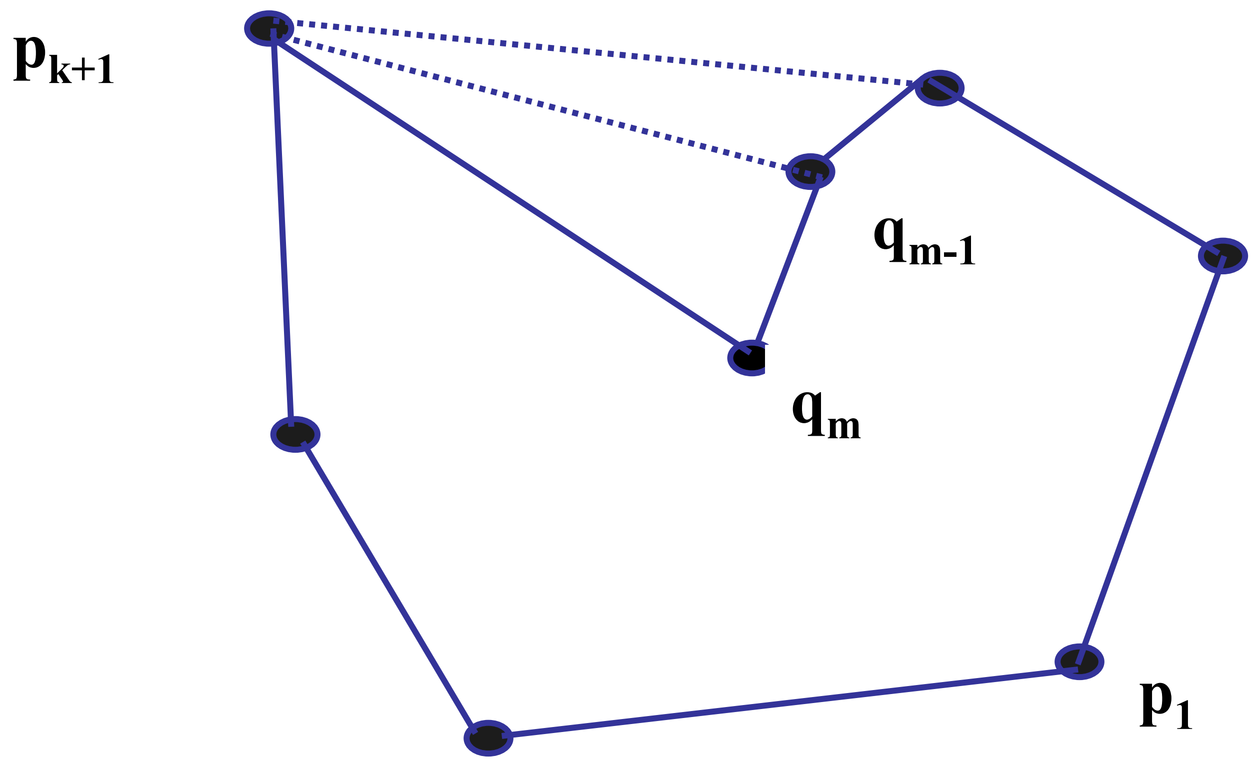
Procedure Gift_Wrapping($p_1, p_2, \ldots p_n$)

Input : $p_1, p_2, \ldots p_n$ (a set of points in the plane)
Output : P (the convex hull of $p_1, p_2, \ldots p_n$)

1. $P \leftarrow \{0\}$ or ε;
2. $p \leftarrow$ a point in the set with the largest x-coordinate;
3. Add p to P;
4. $L \leftarrow$ line containing p and parallel to the x-axis;
5. **while** $|P| < n$ **do**
6. $q \leftarrow$ point such that the angle between the line -p-q- and L is minimal among all points;
7.   add q to P;
8. $L \leftarrow$ line -p-q-;
9. $p \leftarrow q$;

Graham's Scan:

Given a set of n points in the plane, ordered according to the algorithm Simple Polygon, we can find a convex path among the first k points whose corresponding convex polygon encloses the first k points.

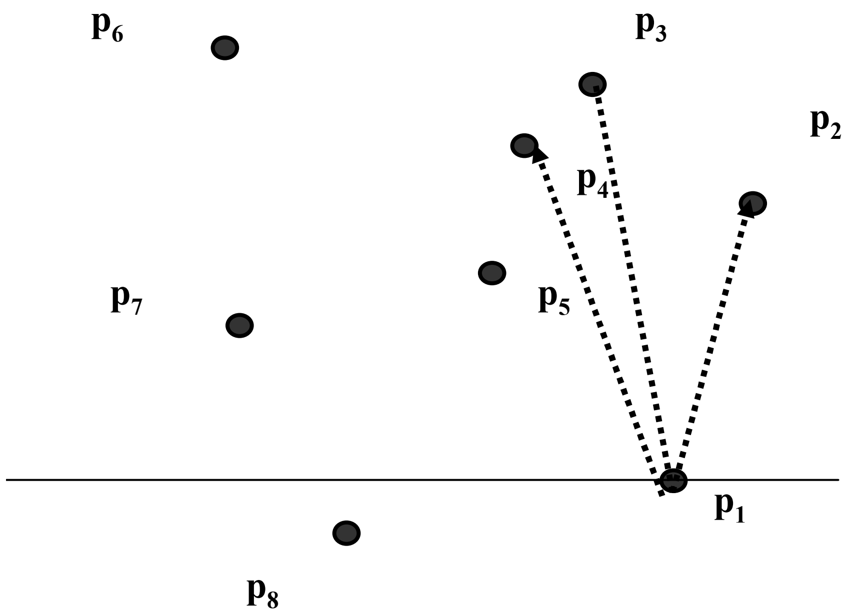


Procedure Graham's Scan($p_1, p_2, ... p_n$)

Input : $p_1, p_2, ... p_n$(a set of points in the plane)
Output : $q_1, q_2, ... q_n$(the convex hull of $p_1, p_2, ... p_n$)

p1 $\leftarrow$ the point in the set with the largest x-coordinate
(and smallest y-coordinate if there are more than one point with the same x-coordinate)
Construct Simple Polygon and arrange points in order
Let order be $p_1, p_2, ... p_n$
$q_1 \leftarrow p_1$;
$q_2 \leftarrow p_2$;
$q_3 \leftarrow p_3$; (initially P consists of p_1, p_2,and p_3)
$m \leftarrow 3$;
for $k \leftarrow 4$ **to** n **do**
  while the angle between lines $-q_{m-1}-q_m-$ and $-q_m-p_k- \geq 180^o$ **do**
    $m \leftarrow m-1$;
  $m \leftarrow m+1$;
  $q_m \leftarrow p_k$;

[Internal to the polygon]

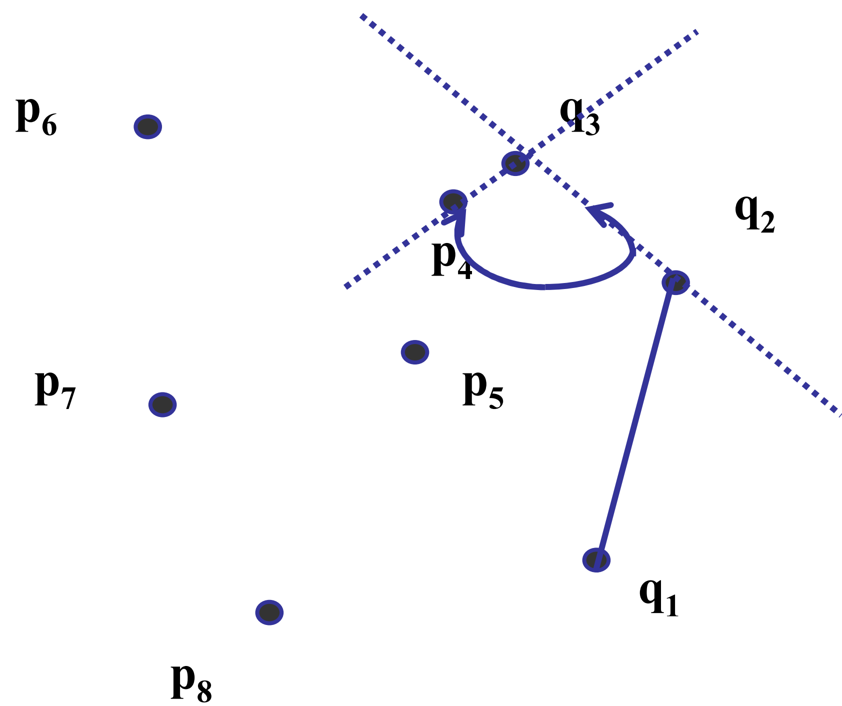


q1:p1

q2:p2

q3:p3

m:3

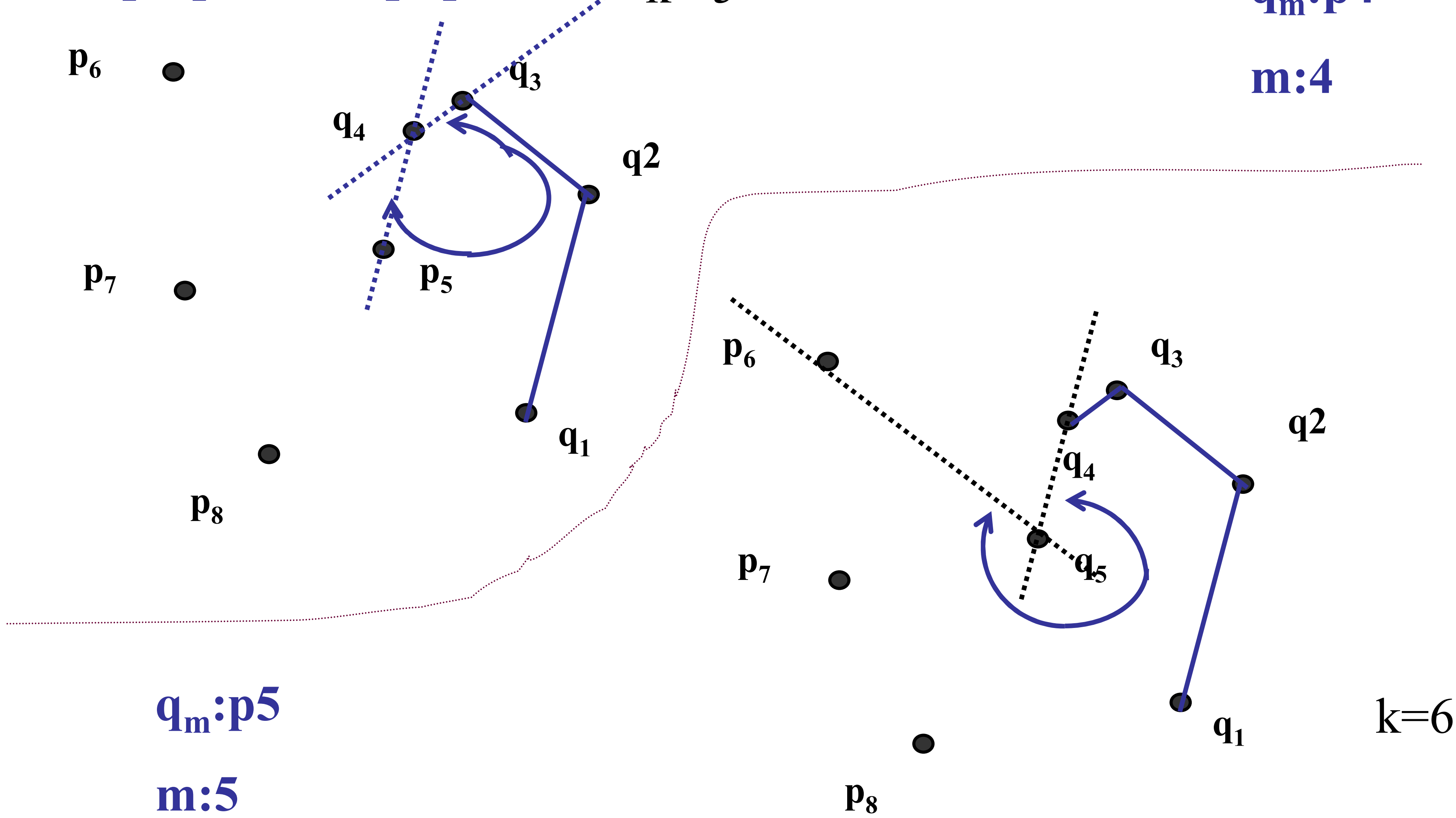


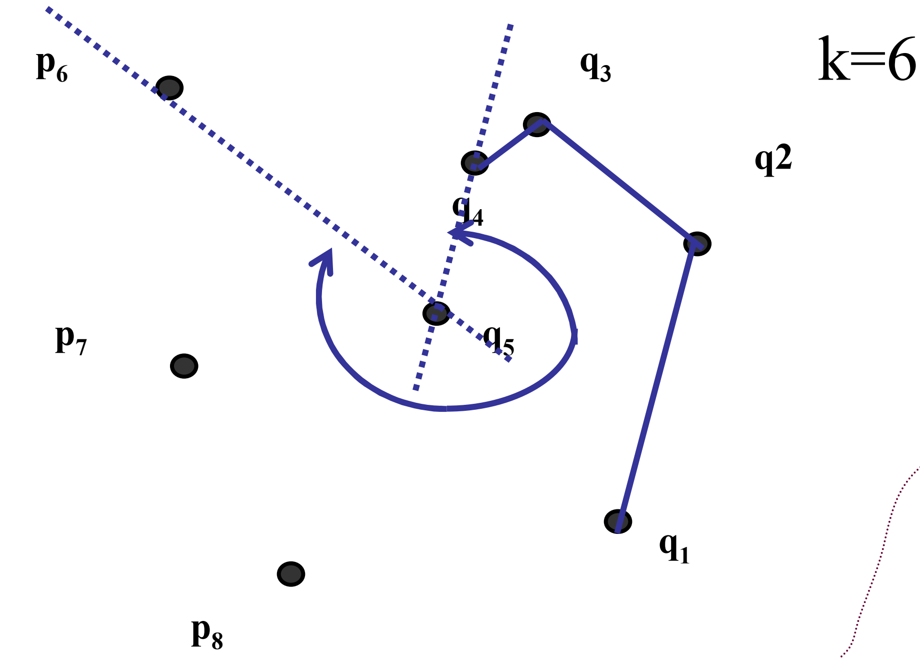
-q2-q3- and -q3-p4-

The angle is < 180

k =4

-q3-q4- and -q4-p5-





Angle between -q4-q5- and -q5-p6- is greater than 180

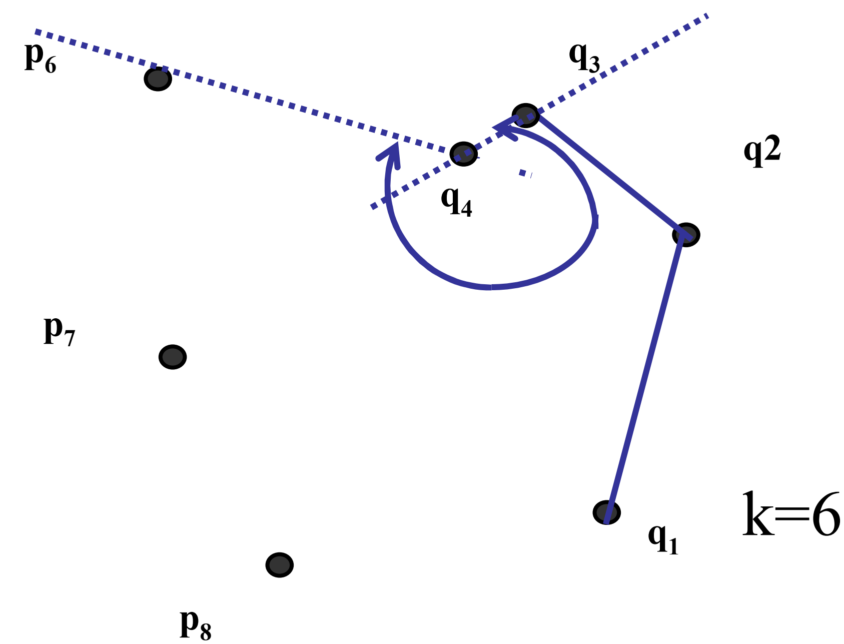
Therefore m = m-1 = 4

We skip p5

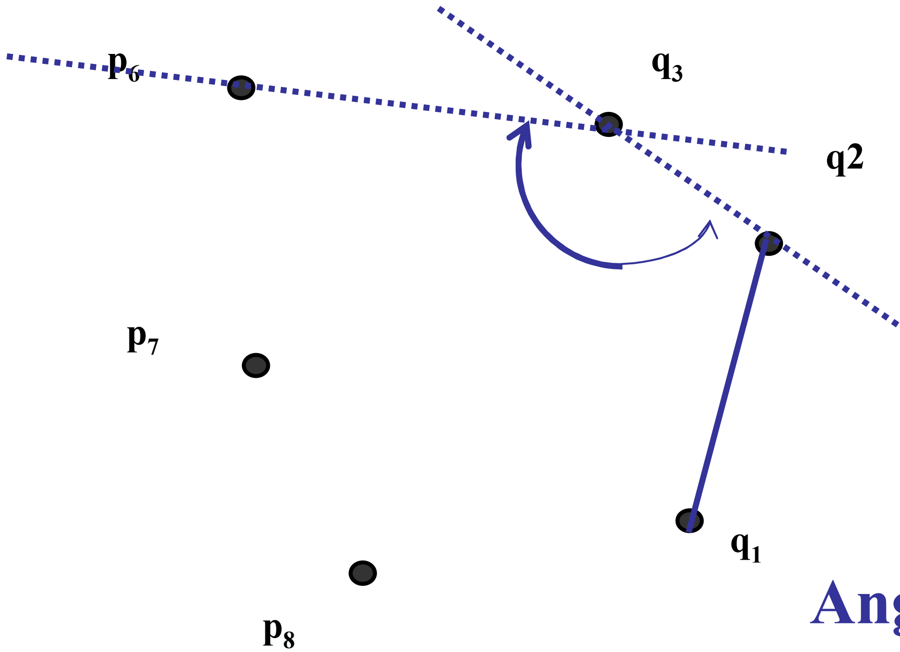
Angle between -q3-q4- and -q4-p6- is greater than 180

Therefore m = m-1 = 3

We skip p4



-q3-q4- and -q4-p6-

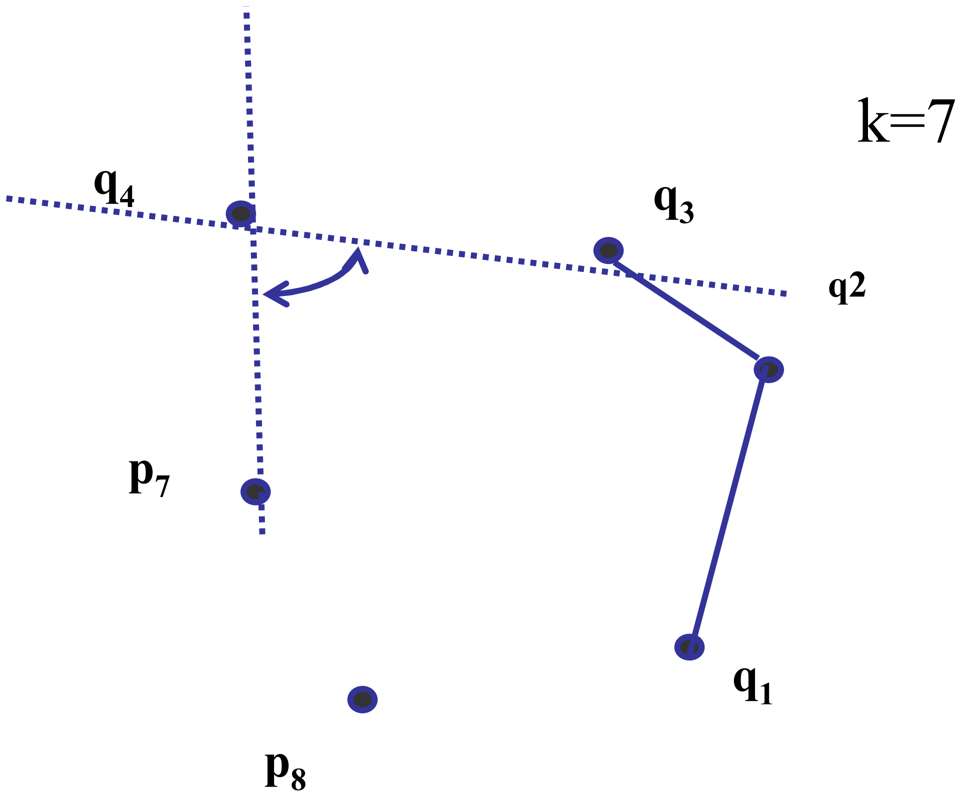


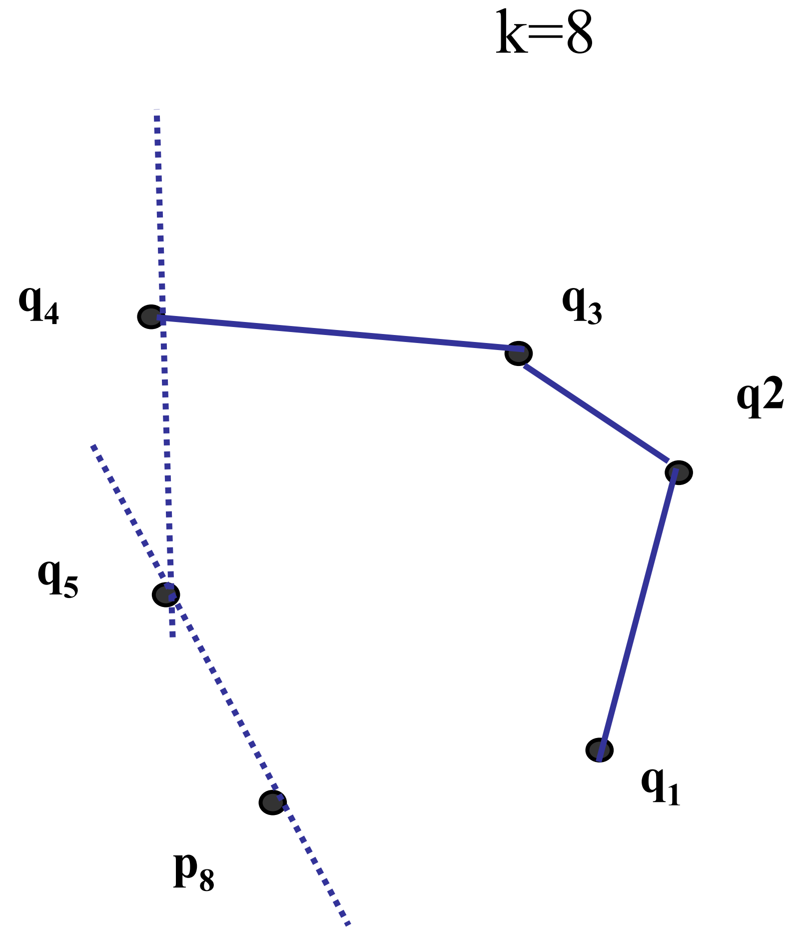
k=6

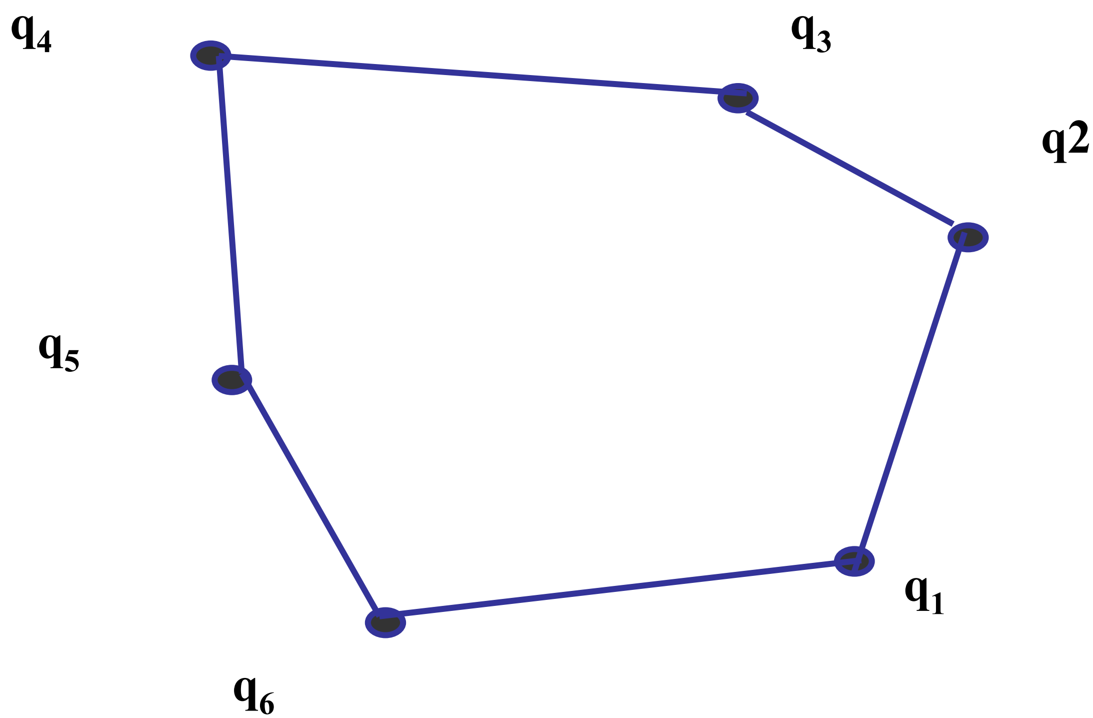
Angle between -q2-q3- and -q3-p6- is less than 180

Therefore m = m+1 = 4

and q4 = p6


k=7
q4
q3
q2
p7
q1
p8


k=8
q4
q3
q2
q5
q1
p8


q_4
q_3
q2
q_5
q_1
q_6

Procedure Graham's Scan(p_1,p_2, . . . p_n)

Input : p_1,p_2, . . . p_n(a set of points in the plane)
Output : q_1,q_2, . . . q_n(the convex hull of p_1,p_2, . . . p_n)

p1 $\leftarrow$ the point in the set with the largest x-coordinate
(and smallest y-coordinate if there are more than one point with the same x-coordinate)
Construct Simple Polygon and arrange points in order
Let order be p_1,p_2, . . . p_n
$q_1 \leftarrow p_1$;
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    while the angle between lines $-q_{m-1}-q_m-$ and $-q_m-p_k-$ $\geq 180^o$ **do**
        $m \leftarrow m-1$;
    $m \leftarrow m+1$;
    $q_m \leftarrow p_k$;

[Internal to the polygon]

Exercise Problems

1. **Let *P* be a simple (not necessarily convex) polygon enclosed in a given rectangle *R*, and *q* be an arbitrary point inside *R*. Design an efficient algorithm to find a line segment connecting *q* to any point outside *R* such that the number of edge of *P* that this line intersects is minimum.**
2. **Let *P* be a set of *n* points in a plane. We define the depth of a point *p* in *P* as the number of convex hulls that need to be 'peeled' (removed) for *p* to become a vertex of the convex hull. Design an $O(n^2)$ algorithm to find the depths of all points in *P*.**
3. **Given a set of n points in the plane *P*. A straight forward or brute force algorithm will take $O(n^2)$ to compute a pair of closest points. Give an $O(n\log^2 n)$ algorithm find a pair of closest points. You get a bonus if you can give an $O(n \log n)$ algorithm**