Analysis of Algorithms

CSE 2320 – Algorithms and Data Structures
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Analysis of Algorithms

- Given an algorithm, some key questions to ask are:
  - How efficient is this algorithm?
  - Can we predict its running time on specific inputs?
  - Should we use this algorithm or should we use an alternative?
  - Should we try to come up with a better algorithm?
- Chapter 2 establishes some guidelines for answering these questions.
- Using these guidelines, sometimes we can obtain easy answers.
  - At other times, getting the answers may be more difficult.

Empirical Analysis

- This is an alternative to the more mathematically oriented methods we will consider.
- Running two alternative algorithms on the same data and comparing the running times can be a useful tool.
  - 1 second vs. one minute is an easy-to-notice difference.
- However, sometimes empirical analysis is not a good option.
  - For example, if it would take days or weeks to run the programs.

Data for Empirical Analysis

- How do we choose the data that we use in the experiments?

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  - Actual data.
    - Pros:
    - Cons:
  - Random data.
    - Pros:
    - Cons:
  - Perverse data.
    - Pros:
    - Cons:

Data for Empirical Analysis

- How do we choose the data that we use in the experiments?
  - Actual data.
    - Pros: give the most relevant and reliable estimates of performance.
    - Cons: may be hard to obtain.
  - Random data.
    - Pros: easy to obtain, make the estimate not data-specific.
    - Cons: may be too unrealistic.
  - Perverse data.
    - Pros: gives us worst case estimate, so we can obtain guarantees of performance.
    - Cons: the worst case estimate may be much worse than average performance.

Comparing Running Times

- When comparing running times of two implementations, we must make sure the comparison is fair.
- We are often much more careful optimizing "our" algorithm compared to the "competitor" algorithm.
- Implementations using different programming languages may tell us more about the difference between the languages than the difference between implementations.
- An easier case is when both implementations use mostly the same codebase, and differ in a few lines.
  - Example: the different implementations of Union-Find in Chapter 1.

Avoid Insufficient Analysis

- Not performing analysis of algorithmic performance can be a problem.
  - Many (perhaps the majority) of programmers have no background in algorithms.
  - People with background in algorithmic analysis may be too lazy, or too pressured by deadlines, to use this background.
- Unnecessarily slow software is a common consequence when skipping analysis.

Avoid Excessive Analysis

- Worrying too much about algorithm performance can also be a problem.
  - Sometimes, slow is fast enough.
  - A user will not even notice an improvement from a millisecond to a microsecond.
  - The time spent optimizing the software should never exceed the total time saved by these optimizations.
    - E.g., do not spend 20 hours to reduce running time by 5 hours on a software that you will only run 3 times and then discard.
- Ask yourself: what are the most important bottlenecks in my code, that I need to focus on?
- Ask yourself: is this analysis worth it? What do I expect to gain?

Mathematical Analysis of Algorithms

- Some times it may be hard to mathematically predict how fast an algorithm will run.
- However, we will study a relatively small set of techniques that applies on a relatively broad range of algorithms.
- First technique: find <u>**key operations**</u> and <u>**key quantities**</u>.
  - Identify the important **operations in the program** that constitute the bottleneck in the computations.
    - This way, we can focus on estimating the number of times these operations are performed, vs. trying to estimate the number of CPU instructions and/or nanoseconds the program will take.
  - Identify a few key quantities that measure the **size of the data** that determine the running time.

Finding Key Operations

- We said it is a good idea to identify the important **operations in the code**, that constitute the bottleneck in the computations.
- How can we do that?

Finding Key Operations

- We said it is a good idea to identify the important **operations in the code**, that constitute the bottleneck in the computations.
- How can we do that?
  - One approach is to just think about it.
  - Another approach is to use software profilers, which show how much time is spent on each line of code.

Finding Key Operations

- What were the key operations for Union Find?
  - ???
- What were the key operations for Binary Search?
  - ???
- What were the key operations for Selection Sort?
  - ???

Finding Key Operations

- What were the key operations for Union Find?
  - Checking and changing ids in **Find.**
  - Checking and changing ids in **Union**.
- What were the key operations for Binary Search?
  - Comparisons between numbers.
- What were the key operations for Selection Sort?
  - Comparisons between numbers.
- In all three cases, the running time was proportional to the total number of those key operations.

Finding Key Quantities

- We said that it is a good idea to identify a few key quantities that measure the **size of the data** and that are the most important in determining the running time.
- What were the key quantities for Union-Find?
  - ???
- What were the key quantities for Binary Search?
  - ???
- What were the key quantities for Selection Sort?
  - ???

Finding Key Quantities

- We said that it is a good idea to identify a few key quantities that measure the **size of the data** and that are the most important in determining the running time.
- What were the key quantities for Union-Find?
  - Number of nodes, number of edges.
- What were the key quantities for Binary Search?
  - Size of the array.
- What were the key quantities for Selection Sort?
  - Size of the array.

Finding Key Quantities

- These key quantities are different for each set of data that the algorithm runs on.
- Focusing on these quantities greatly simplifies the analysis.
  - For example, there is a huge number of integer arrays of size 1,000,000, that could be passed as inputs to Binary Search or to Selection Sort.
  - However, to analyze the running time, we do not need to worry about the **contents** of these arrays (which are too diverse), but just about the **size**, which is expressed as a single number.

Describing Running Time

- Rule: most algorithms have a primary parameter **N**, that measures the size of the data and that affects the running time most significantly.
- Example: for binary search, **N** is ???
- Example: for selection sort, **N** is ???
- Example: for Union-Find, **N** is ???

Describing Running Time

- Rule: most algorithms have a primary parameter **N**, that measures the size of the data and that affects the running time most significantly.
- Example: for binary search, **N** is the size of the array.
- Example: for selection sort, **N** is the size of the array.
- Example: for Union-Find, **N** is ???
  - Union-Find is one of many exceptions.
  - Two key parameters, number of nodes, and number of edges, must be considered to determine the running time.

Describing Running Time

- Rule: most algorithms have a primary parameter **N**, that affects the running time most significantly.
- When we analyze an algorithm, our goal is to find a function **f(N)**, such that the running time of the algorithm is **proportional** to **f(N)**.
- Why **proportional** and not **equal**?

Describing Running Time

- Rule: most algorithms have a primary parameter **N**, that affects the running time most significantly.
- When we analyze an algorithm, our goal is to find a function **f(N)**, such that the running time of the algorithm is **proportional** to **f(N)**.
- Why **proportional** and not **equal**?
- Because **the actual running time is not a defining characteristic of an algorithm**.
  - Running time depends on programming language, actual implementation, compiler used, machine executing the code, …

Describing Running Time

- Rule: most algorithms have a primary parameter **N**, that affects the running time most significantly.
- When we analyze an algorithm, our goal is to find a function **f(N)**, such that the running time of the algorithm is **proportional** to **f(N)**.
- We will now take a look at the most common functions that are used to describe running time.

The Constant Function: **f(N) = 1**

- **f(N) = 1**. What does it mean to say that the running time of an algorithm is described by **1**?

The Constant Function: **f(N) = 1**

- **f(N) = 1**. What does it mean to say that the running time of an algorithm is described by **1**?
- It means that the running time of the algorithm is proportional to **1**, which means…

The Constant Function: **f(N) = 1**

- **f(N) = 1**: What does it mean to say that the running time of an algorithm is described by **1**?
- It means that the running time of the algorithm is proportional to **1**, which means…
  - that the running time is *constant*, or at least bounded by a constant.
- This happens when all instructions of the program are executed only once, or at least no more than a certain fixed number of times.
- If **f(N) = 1**, we say that **the algorithm takes constant time**. This is the best case we can ever hope for.

The Constant Function: **f(N) = 1**

- What algorithm (or part of an algorithm) have we seen whose running time is constant?

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- What algorithm (or part of an algorithm) have we seen whose running time is constant?
- The **find** operation in the quick-find version of Union-Find.

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- How good or bad is that?

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- How good or bad is that?
  - log 1000 ~= ???.
  - The logarithm of one million is about ???.
  - The logarithm of one billion is about ???.
  - The logarithm of one trillion is about ???.

Logarithmic Time: **f(N) = log N**

- **f(N) = log N**: the running time is proportional to the logarithm of N.
- How good or bad is that?
  - log 1000 ~= 10.
  - The logarithm of one million is about 20.
  - The logarithm of one billion is about 30.
  - The logarithm of one trillion is about 40.
- Function **log N** grows *very* slowly:
- This means that the running time when N = one trillion is only **four times** the running time when N = 1000. This is **really good** scaling behavior.

Logarithmic Time: **f(N) = log N**

- If **f(N) = log N**, we say that the algorithm takes **logarithmic time**.
- What algorithm (or part of an algorithm) have we seen whose running time is proportional to **log N?**

Logarithmic Time: **f(N) = log N**

- If **f(N) = log N**, we say that the algorithm takes **logarithmic time**.
- What algorithm (or part of an algorithm) have we seen whose running time is proportional to **log N?**
- Binary Search.
- The **Find** function on the weighted-cost quick-union version of Union-Find.

Logarithmic Time: **f(N) = log N**

- Logarithmic time commonly occurs when solving a big problem is solved in a sequence of steps, where:
  - Each step reduces the size of the problem by some constant factor.
  - Each step requires no more than a constant number of operations.
- Binary search is an example:
  - Each step reduces the size of the problem by a factor of 2.
  - Each step requires only one comparison, and a few variable updates.

Linear Time: **f(N) = N**

- **f(N) = N**: the running time is proportional to N.
- This happens when we need to do some fixed amount of processing on each input element.
- What algorithms (or parts of algorithms) are examples?

Linear Time: **f(N) = N**

- **f(N) = N**: the running time is proportional to N.
- This happens when we need to do some fixed amount of processing on each input element.
- What algorithms (or parts of algorithms) are examples?
  - The **Union** function in the quick-find version of Union-Find.
  - Sequential search for finding the min or max value in an array.
  - Sequential search for determining whether a value appears somewhere in an array.
    - Is this ever useful? Can't we always just do binary search?

Linear Time: **f(N) = N**

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  - The **Union** function in the quick-find version of Union-Find.
  - Sequential search for finding the min or max value in an array.
  - Sequential search for determining whether a value appears somewhere in an array.
    - Is this ever useful? Can't we always just do binary search?
    - If the array is not already sorted, binary search does not work.

N log N Time

- **f(N) = N log N**: the running time is proportional to N log N.
- This running time is commonly encountered, especially in algorithms working as follows:
  - Break problem into smaller subproblems.
  - Solve subproblems independently.
  - Combine the solutions of the subproblems.
- Many sorting algorithms have this complexity.
- Comparing linear to **N log N** time.
  - N = 1 million, N log N is about ???
  - N = 1 billion, N log N is about ???
  - N = 1 trillion, N log N is about ???

N log N Time

- Comparing linear to **N log N** time.
  - N = 1 million, N log N is about 20 million.
  - N = 1 billion, N log N is about 30 billion.
  - N = 1 trillion, N log N is about 40 trillion.
- **N log N** is worse than linear time, but **not by much**.

Quadratic Time

- **f(N) = $\mathbf{N^2}$**: the running time is proportional to the square of **N**.
- In this case, we say that the running time is **quadratic** to **N**.
- Any example where we have seen quadratic time?

Quadratic Time

- **f(N) = $\mathbf{N^2}$**: the running time is proportional to the square of **N**.
- In this case, we say that the running time is **quadratic** to **N**.
- Any example where we have seen quadratic time?
  – Selection Sort.

Quadratic Time

- Comparing linear, **N log N**, and quadratic time.

N	N log N	N^2
10^6 (1 million)	about 20 million	10^{12} (one trillion)
10^9 (1 billion)	about 30 billion	10^{18} (one quintillion)
10^{12} (1 trillion)	about 40 trillion	10^{24} (one septillion)

- Quadratic time algorithms become impractical (too slow) much faster than linear and **N log N** time algorithms.
- Of course, what we consider "impractical" depends on the application.
  - Some applications are more tolerant of longer running times.

Cubic Time

- **f(N) = N^3**: the running time is proportional to the cube of **N**.
- In this case, we say that the running time is **cubic** to **N**.

Cubic Time

- Example of a problem whose solution has cubic running time: the assignment problem.
  - We have two sets **A** and **B**. Each set contains **N** items.
  - We have a cost function **C(a, b)**, assigning a cost to matching an item **a** of **A** with an item **b** of **B**.
  - Find the optimal one-to-one correspondence (i.e., a way to match each element of A with one element of B and vice versa), so that the sum of the costs is minimized.

Cubic Time

- Wikipedia example of the assignment problem:
  - We have three workers, Jim, Steve, and Alan.
  - We have three jobs that need to be done.
  - There is a different cost associated with each worker doing each job.

	Clean bathroom	Sweep floors	Wash windows
Jim	\$1	\$3	\$3
Steve	\$3	\$2	\$3
Alan	\$3	\$4	\$2

  - What is the optimal job assignment?
- Cubic running time means that it is too slow to solve this problem for, let's say, N = 1 million.

Exponential Time

- **$f(N) = 2^N$**: this is what we call **exponential running time**.
- Such algorithms are usually too slow unless **N** is small.
- Even for **N** = 100, **2^N** is too large and the algorithm will not terminate in our lifetime, or in the lifetime of the Universe.
- Exponential time arises when we try all possible combinations of solutions.
  – Example: travelling salesman problem: find an itinerary that goes through each of **N** cities, visits no city twice, and minimizes the total cost of the tickets.
- Quantum computers (if they ever arrive) may solve <u>**some**</u> of these problems with manageable running time.

Some Useful Constants and Functions

symbol	value
e	2.71828…
γ (gamma)	0.57721…
ϕ (phi)	$(1 + \sqrt{5}) / 2 = 1.61803...$

These tables are for reference. We may use such symbols and functions as we discuss specific algorithms.

function	name	approximation
$\lfloor x \rfloor$	floor function	x
$\lceil x \rceil$	ceiling function	x
F_N	Fibonacci numbers	$\phi^N / \sqrt{5}$
H_N	harmonic numbers	$\ln(N) + \gamma$
N!	factorial function	$(N / e)^N$
lg(N!)		N lg(N) - 1.44N

Motivation for Big-Oh Notation

- Given an algorithm, we want to find a function that describes the running time of the algorithm.
- Key question: how much data can this algorithm handle in a reasonable time?
- There are some details that we would actually **NOT** want this function to include, because they can make a function unnecessarily complicated.
  - Constants.
  - Behavior fluctuations on small data.
- The Big-Oh notation, which we will see in a few slides, achieves that, and greatly simplifies algorithmic analysis.

Why Constants Are Not Important

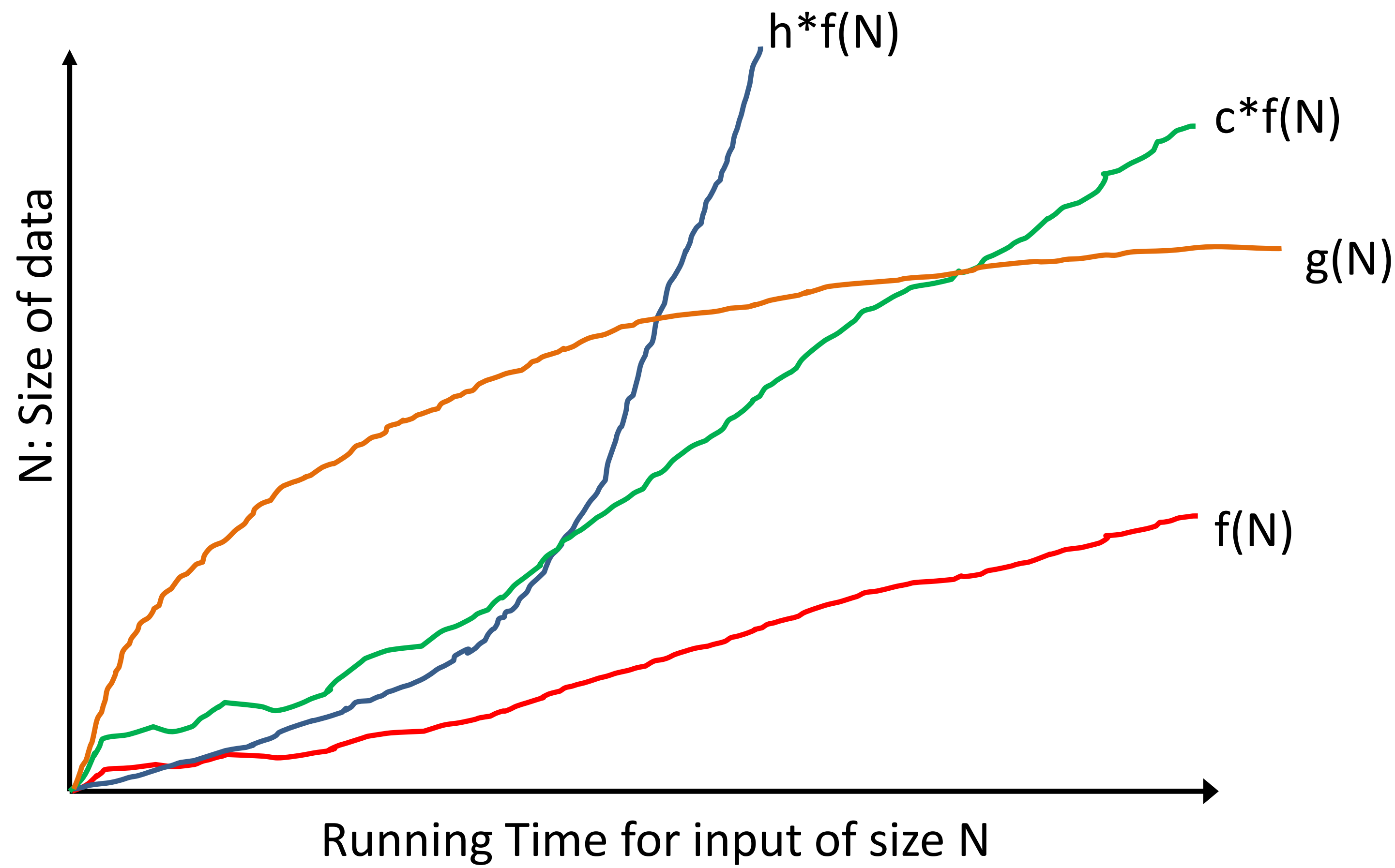
- Does it matter if the running time is f(N) or 5*f(N)?

Why Constants Are Not Important

- Does it matter if the running time is f(N) or 5*f(N)?
- For the purposes of algorithmic analysis, it typically does NOT matter.
- Constant factors are NOT an inherent property of the algorithm. They depend on parameters that are independent of the algorithm, such as:
  - Choice of programming language.
  - Quality of the code.
  - Choice of compiler.
  - Machine capabilities (CPU speed, memory size, …)

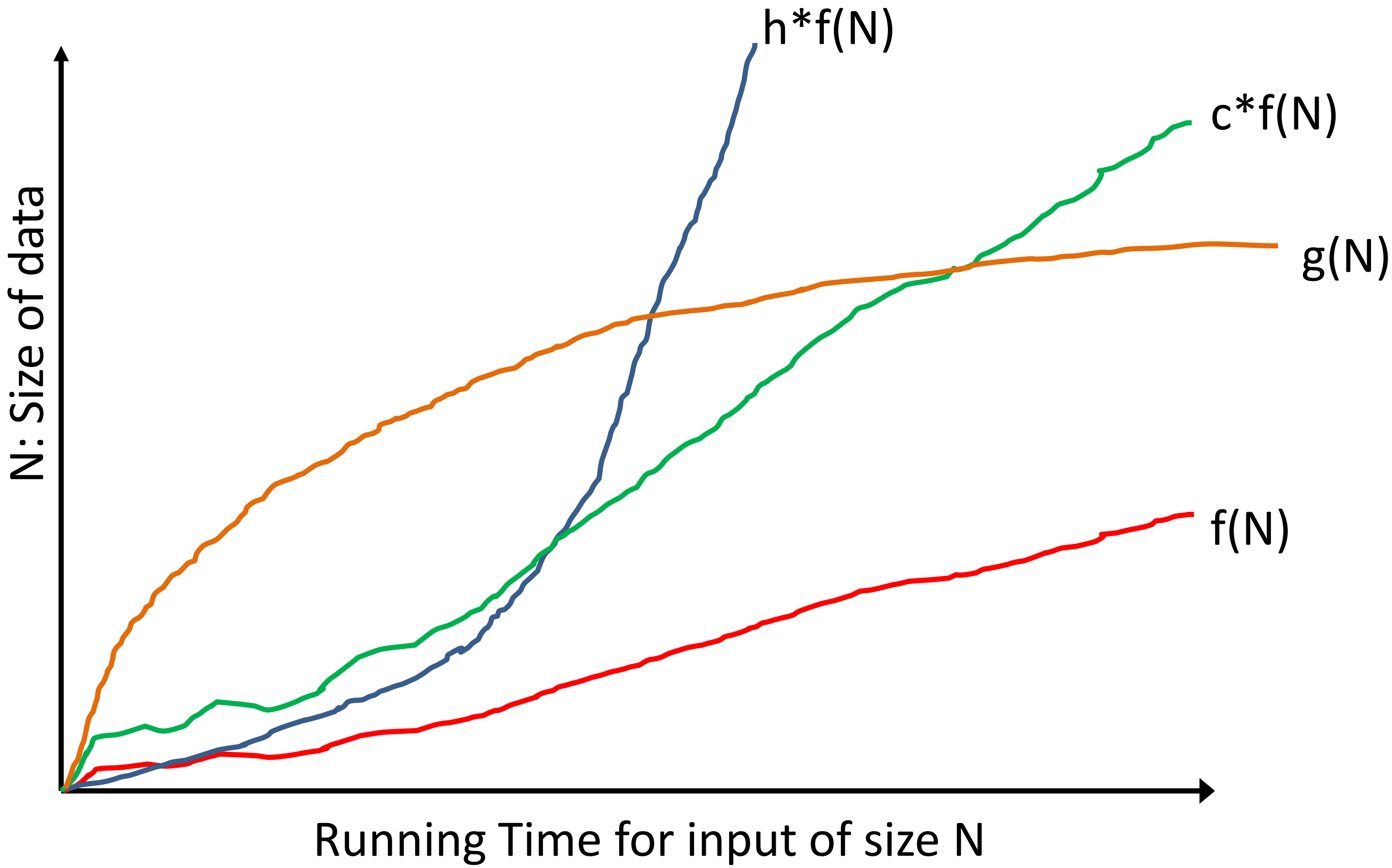
Why Asymptotic Behavior Matters

- Asymptotic behavior: The behavior of a function as the input approaches infinity.



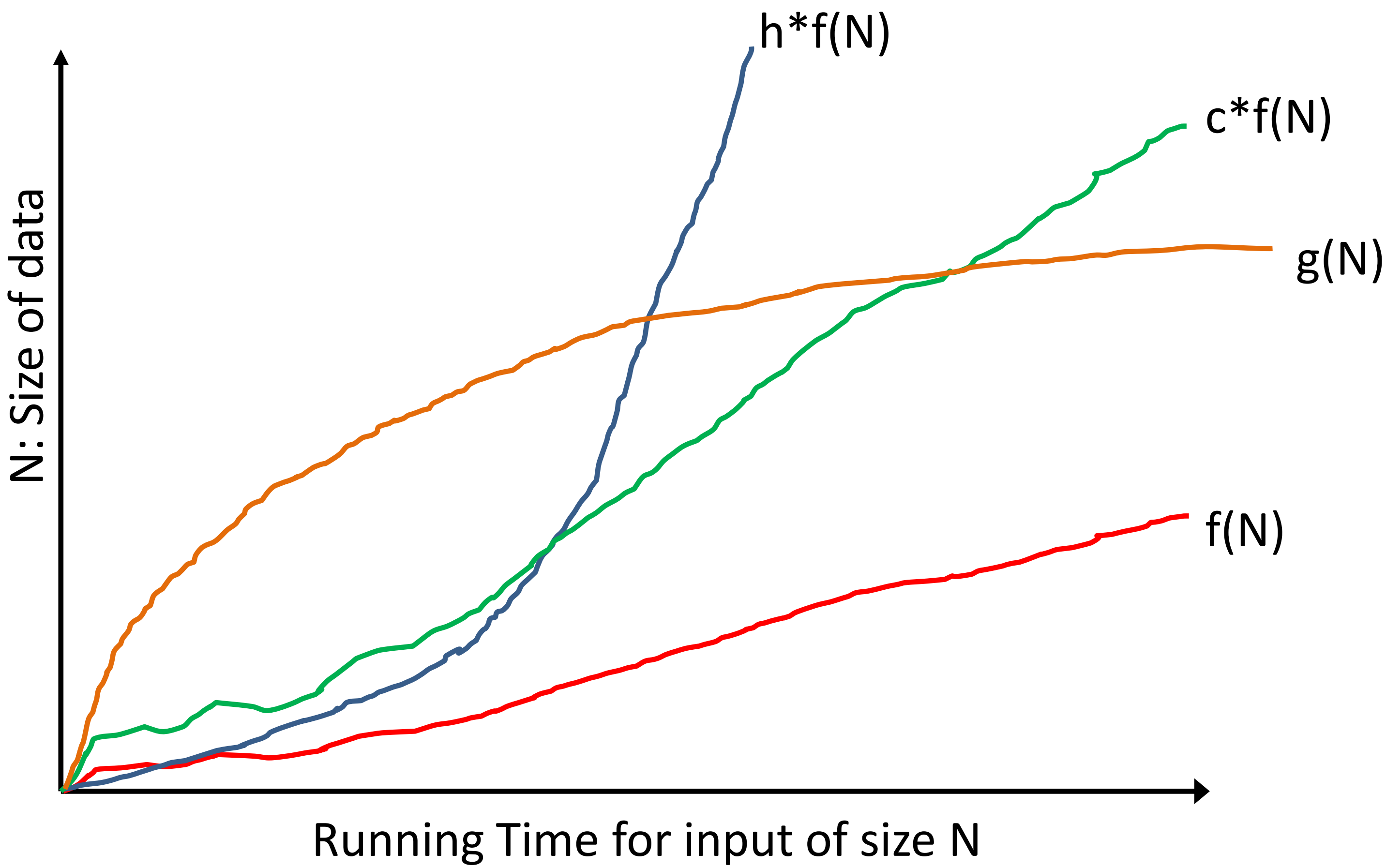
Why Asymptotic Behavior Matters

- Which of these functions works best asymptotically?



Why Asymptotic Behavior Matters

- Which of these functions works best asymptotically?
  - g(N) seems to grow VERY slowly after a while.



Big-Oh Notation

- A function *g(N)* is said to be *O(f(N))* if there exist constants c_0 and N_0 such that:

  $g(N) < c_0 f(N)$ for all $N > N_0$.

- THIS IS THE SINGLE MOST IMPORTANT THING YOU LEARN IN THIS COURSE.
- Typically, *g(N)* is the running time of an algorithm, in your favorite units, implementation, and machine. This can be a rather complicated function.
- In algorithmic analysis, we try to find a *f(N)* that is **simple**, and such that *g(N) = O(f(N))*.

Why Use Big-Oh Notation?

- A function $g(N)$ is said to be $O(f(N))$ if there exist constants c_0 and N_0 such that:

  $g(N) < c_0 f(N)$ for all $N > N_0$.

- The Big-Oh notation greatly simplifies the analysis task, by:
  1. Ignoring constant factors. How is this achieved?
     - By the c_0 in the definition. We are free to choose ANY constant c_0 we want, to make the formula work.
     - Thus, Big-Oh notation is independent of programming language, compiler, machine performance, and so on…

Why Use Big-Oh Notation?

- A function *g(N)* is said to be *O(f(N))* if there exist constants c_0 and N_0 such that:

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- The Big-Oh notation greatly simplifies the analysis task, by:
  2. Ignoring behavior for small inputs. How is this achieved?
     - By the N_0 in the implementation. If a finite number of values are not compatible with the formula, just ignore them.
     - Thus, big-Oh notation focuses on asymptotic behavior.

Why Use Big-Oh Notation?

- A function $g(N)$ is said to be $O(f(N))$ if there exist constants c_0 and N_0 such that:

  $g(N) < c_0 f(N)$ for all $N > N_0$.

- The Big-Oh notation greatly simplifies the analysis task, by:
  3. Allowing us to describe complex running time behaviors of complex algorithms with simple functions, such as N, log N, N^2, 2^N, and so on.
     - Such simple functions are sufficient for answering many important questions, **once you get used to Big-Oh notation**.

Inferences from Big-Oh Notation

- Binary search takes logarithmic time.
- This means that, if $g(N)$ is the running time, there exist constants c_0 and N_0 such that:

$g(N) < c_0 \log(N)$ for all $N > N_0$.

- Can this function handle trillions of data in reasonable time?
  - NOTE: the question is about **time**, not about **memory**.

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- Can this function handle trillions of data in reasonable time?
  - NOTE: the question is about time, not about memory.
- The answer is an easy YES!
  - We don't even know what c_0 and N_0 are, and we don't care.
  - The key thing is that the running time is O(log(N)).

Inferences from Big-Oh Notation

- Selection Sort takes quadratic time.
- This means that, if $g(N)$ is the running time, there exist constants c_0 and N_0 such that:

  $g(N) < c_0 N^2$ for all $N > N_0$.

- Can this function handle one billion data in reasonable time?

Inferences from Big-Oh Notation

- Selection Sort takes quadratic time.
- This means that, if $g(N)$ is the running time, there exist constants c_0 and N_0 such that:

  $g(N) < c_0 N^2$ for all $N > N_0$.

- Can this function handle one billion data in reasonable time?
- The answer is an easy NO!
  - Again, we don't know what c_0 and N_0 are, and we don't care.
  - The key thing is that the running time is quadratic.

Is Big-Oh Notation Always Enough?

- NO! Big-Oh notation does not always tell us which of two algorithms is preferable.

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- NO! Big-Oh notation does not always tell us which of two algorithms is preferable.
  - Example 1: if we know that the algorithm will only be applied to relatively small N, we may prefer a running time of N^2 nanoseconds over *log(N)* centuries.
  - Example 2: even constant factors can be important. For many applications, we strongly prefer a running time of *3N* over *1500N*.

Is Big-Oh Notation Always Enough?

- NO! Big-Oh notation does not always tell us which of two algorithms is preferable.
  - Example 1: if we know that the algorithm will only be applied to relatively small N, we may prefer a running time of N^2 nanoseconds over *log(N)* centuries.
  - Example 2: even constant factors can be important. For many applications, we strongly prefer a running time of *3N* over *1500N*.
- Big-Oh notation is not meant to tells us everything about running time.
- But, Big-Oh notation tells us a lot, and is often much easier to compute than actual running times.

Simplifying Big-Oh Notation

- Suppose that we are given this running time:
  $g(N) = 35N^2 + 41N + log(N) + 1532.$
- How can we express g(N) in Big-Oh notation?

Simplifying Big-Oh Notation

- Suppose that we are given this running time: $g(N) = 35N^2 + 41N + log(N) + 1532.$
- How can we express g(N) in Big-Oh notation?
- Typically we say that $g(N) = O(N^2).$
- The following are also correct, but unnecessarily complicated, and thus less useful, and rarely used.
  - $g(N) = O(N^2) + O(N).$
  - $g(N) = O(N^2) + O(N) + O(logN) + O(1).$
  - $g(N) = O(35N^2 + 41N + log(N) + 1532).$

Simplifying Big-Oh Notation

- Suppose that we are given this running time: $g(N) = 35N^2 + 41N + \log(N) + 1532$.
- We say that $g(N) = O(N^2)$.
- Why is this mathematically correct?
  - Why can we ignore the non-quadratic terms?
- This is where the Big-Oh definition comes into play. We can find an N_0 such that, for all $N > N_0$: $g(N) < 36N^2$.
  - If you don't believe this, do the calculations for practice.

Simplifying Big-Oh Notation

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- Another way to show correctness: as *N* goes to infinity, what is the limit of $g(N) / N^2$?

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- Why is this mathematically correct?
  - Why can we ignore the non-quadratic terms?
- Another way to show correctness: as *N* goes to infinity, what is the limit of $g(N) / N^2$?
  - 35.
  - This shows that the non-quadratic terms become negligible as *N* gets larger.

Trick Question

- Let $g(N) = N \log N$.
- Is it true that $g(N) = O(N^{100})$?

Trick Question

- Let $g(N) = N \log N$.
- Is it true that $g(N) = O(N^{100})$?
- Yes. Let's look again at the definition of Big-Oh:
- A function $g(N)$ is said to be $O(f(N))$ if there exist constants c_0 and N_0 such that:

  $g(N) < c_0 f(N)$ for all $N > N_0$.

- Note the "<" sign to the right of $g(N)$.
- Thus, if $g(N) = O(f(N))$ and $f(N) < h(N)$, it follows that $g(N) = O(h(N))$.

Omega (Ω) and Theta (Θ) Notations

- If $f(N) = O(g(N))$, then we also say that $g(N) = \Omega(f(N))$.
- If $f(N) = O(g(N))$ **and** $f(N) = \Omega(g(N))$, then we say that $f(N) = \Theta(g(N))$.
- The Theta notation is clearly stricter than the Big-Oh notation:
  - We can say that $N^2 = O(N^{100})$.
  - We **cannot** say that $N^2 = \Theta(N^{100})$.

Big-Oh vs. Theta (Θ) Notation

- The Big-Oh notation indicates a worst-case bound.
- However, oftentimes (almost all the time) we say $g(N) = O(f(N))$ when we could say that $g(N) = \Theta(f(N))$.
- This is not wrong, it is simply not as strict as it could be. Still, in practice we use Big-Oh notation much more frequently than we use Θ notation.
  - This is somewhat similar to saying $x <= 3$ when we actually know that $x = 3$.

Big-Oh vs. Theta (Θ) Notation

- Why do we use so often Big-Oh instead of Θ? Partly habit, and partly because we care so much about worst cases.
  - E.g., saying that the running time is O(N log N) establishes that the running time is not worse than O(N log N).
- Also, for some problems, coming up with an algorithm that solves the problem in *Θ(f(N))* only means that the problem is solvable in *O(f(N))*. Why?

Big-Oh vs. Theta (Θ) Notation

- Why do we use so often Big-Oh instead of Θ? Partly habit, and partly because we care so much about worst cases.
  - E.g., saying that the running time is O(N log N) establishes that the running time is not worse than O(N log N).
- Also, for some problems, coming up with an algorithm that solves the problem in *Θ(f(N))* only means that the problem is solvable in *O(f(N))*. Why?
  - Because the algorithm we came up with may not be optimal.
  - Unless we prove that our algorithm is optimal, it is mathematically possible that a better algorithm exists, with lower running time.

Using Limits

- if $\lim_{n \to \infty} \frac{g(N)}{f(N)}$ is a constant, then g(N) = O(f(N)).
  - "Constant" includes zero, but does NOT include infinity.

- if $\lim_{n \to \infty} \frac{f(N)}{g(N)} = \infty$ then g(N) = O(f(N)).

- if $\lim_{n \to \infty} \frac{f(N)}{g(N)}$ is a constant, then g(N) = Ω(f(N)).
  - Again, "constant" includes zero, but not infinity.

- if $\lim_{n \to \infty} \frac{f(N)}{g(N)}$ is a **non-zero** constant, then g(N) = Θ(f(N)).
  - In this definition, both zero and infinity are excluded.

Using Limits - Comments

- The previous formulas relating limits to big-Oh notation show once again that big-Oh notation ignores:
  - constants
  - behavior for small values of N.
- How do we see that?
  - In the previous formulas, it is sufficient that the limit is equal to a constant. **The value of the constant does not matter.**
  - In the previous formulas, only **the limit at infinity** matters. This means that we can ignore behavior up to any finite value, if we need to.

Basic Recurrences

- How do we compute the running time of an algorithm in Big-Oh notation?
- Sometimes it is easy, sometimes it is hard.
- We will learn a few simple tricks that work in many cases that we will encounter this semester.

Case 1: Check All Items, Eliminate One

- In this case, the algorithm proceeds in a sequence of similar steps, where:
  - each step loops through all items in the input, and eliminates one item.
- Any examples of such an algorithm?

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- Any examples of such an algorithm?
  - Selection Sort.

Case 1: Check All Items, Eliminate One

- Let g(N) be an approximate estimate of the running time, measured in time units of our convenience.
  - In this case, we choose as time unit the time that it takes to examine one item.
  - Obviously, this is a simplification, since there are other things that such an algorithm will do, in addition to just examining one item.
  - That is one of the plusses of using Big-Oh notation. We can ignore parts of the algorithm that take a relatively small time to run, and focus on the part that dominates running time.

Case 1: Check All Items, Eliminate One

- Let g(N) be the running time.
- Then, g(N) = ???

Case 1: Check All Items, Eliminate One

- Let g(N) be the running time.
- Then, g(N) = g(N-1) + N. Why?
  - Because we need to examine all items (N units of time), and then we need to run the algorithm on N-1 items.
- ```
  g(N) = g(N-1) + N
       = g(N-2) + (N-1) + N
       = g(N-3) + (N-2) + (N-1) + N
       ...
       = 1 + 2 + 3 + ... + (N-1) + N

       = N(N + 1) / 2
  ```
  $= O(N^2)$
- Conclusion: The algorithm takes quadratic time.
  ```

Case 2: Halve the Problem in Constant Time

- In this case, each step of the algorithm consists of:
  - performing a constant number of operations, and then reducing the size of the input by half.
- Any example of such an algorithm?

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- What is a convenient unit of time to use here?

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- Any example of such an algorithm?
  - Binary Search.
- What is a convenient unit of time to use here?
  - The time it takes to do the constant number of operations to halve the input.

Case 2: Halve the Problem in Constant Time

- In this case, each step of the algorithm consists of:
  - performing a constant number of operations, and then reducing the size of the input by half.
- **`g(2`^{n}`) = ???`**

Case 2: Halve the Problem in Constant Time

- In this case, each step of the algorithm consists of:
  - performing a constant number of operations, and then reducing the size of the input by half.
- $$\begin{aligned} g(2^n) &= 1 + g(2^{n-1}) \\ &= 2 + g(2^{n-2}) \\ &= 3 + g(2^{n-3}) \\ &\dots \\ &= n + g(2^0) \\ &= n + 1. \end{aligned}$$
- *O(n)* time for $N = 2^n$.
- Substituting n for log N: *O(log N)* time.

Case 3: Halve the Input in Linear Time

- In this case, each step of the algorithm consists of:
  - Performing a linear (i.e., *O(N)*) number of operations, and then reducing the size of the input by half.
- **`g(N) = ???`**

Case 3: Halve the Input in Linear Time

- In this case, each step of the algorithm consists of:
  - Performing a linear (i.e., *O(N)*) number of operations, and then reducing the size of the input by half.
- ```
  g(N) = g(N/2) + N
       = g(N/4) + N/2 + N
       = g(N/8) + N/4 + N/2 + N
       ...
       = 1 + 2 + 4 + ... + N/4 + N/2 + N
       = ???
  ```
  ```

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     = about 2N
```

- *O(N)* time.

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- What is our unit of time here?

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       ...
       = about 2N
  ```
- What is our unit of time here?
  - *1/N ** time to do the linear number of operations at the first step.
  ```

Case 4: Break Problem Into Two Halves in Linear Time

- In this case, each step of the algorithm consists of:
  - Doing *O(N)* operations to split the problem into two halves.
  - Calling the algorithm recursively on each half.
  - Doing *O(N)* operations to combine the two answers.
- **`g(N) = ???`**

Case 4: Break Problem Into Two Halves in Linear Time

- In this case, each step of the algorithm consists of:
  - Doing *O(N)* operations to split the problem into two halves.
  - Calling the algorithm recursively on each half.
  - Doing *O(N)* operations to combine the two answers.
- **g(N) = 2g(N/2) + N**

```
g(N) = 2g(N/2) + N
     = 4g(N/4) + N + N
     = 8g(N/8) + N + N + N
     ...
     = N log N
```

- What is our unit of time here?

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     ...
     = N log N
```

- What is our unit of time here?
  - *1/N ** time to do the *O(N)* operations at the first step.

Case 4: Break Problem Into Two Halves in Linear Time

- In this case, each step of the algorithm consists of:
  - Doing *O(N)* operations to split the problem into two halves.
  - Calling the algorithm recursively on each half.
  - Doing *O(N)* operations to combine the two answers.
- Note: we have not seen any examples of this case yet, but we will see several such examples when we study sorting algorithms.

Case 5: Break Problem Into Two Halves in Constant Time

- In this case, each step of the algorithm consists of:
  - Doing *O(1)* operations to split the problem into two halves.
  - Calling the algorithm recursively on each half.
  - Doing *O(1)* operations to combine the two answers.
- **`g(N) = ???`**

Case 5: Break Problem Into Two Halves in Constant Time

- In this case, each step of the algorithm consists of:
  - Doing *O(1)* operations to split the problem into two halves.
  - Calling the algorithm recursively on each half.
  - Doing *O(1)* operations to combine the two answers.
- ```
  g(N) = 2g(N/2) + 1
       = 4g(N/4) + 2 + 1
       = 8g(N/8) + 4 + 2 + 1
       ...
       = about N
  ```
- What is our unit of time here?
  ```

Case 5: Break Problem Into Two Halves in Constant Time

- In this case, each step of the algorithm consists of:
  - Doing *O(1)* operations to split the problem into two halves.
  - Calling the algorithm recursively on each half.
  - Doing *O(1)* operations to combine the two answers.
- ```
  g(N) = 2g(N/2) + 1
       = 4g(N/4) + 2 + 1
       = 8g(N/8) + 4 + 2 + 1
       ...
       = about N
  ```
- What is our unit of time here?
  - The time to do the *O(1)* operations at the first step.
  ```