Symbol Tables and Search Trees

CSE 2320 – Algorithms and Data Structures
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Symbol Tables - Dictionaries

- A symbol table is a data structure that allows us to maintain and use an organized set of items.
- Main operations:
  - Insert new item.
  - Search and return an item with a given key.
  - Delete an item.
  - Modify an item.
  - Sort all items.
  - Find k-th smallest item.

Symbol Tables - Dictionaries

- Similarities and differences compared to priority queues:

Symbol Tables - Dictionaries

- Similarities and differences compared to priority queues:
- In priority queues we care about:
  – Insertions.
  – Finding/deleting **the max item** efficiently.
- In symbol tables we care about:
  – Insertions.
  – Finding/deleting **any** item efficiently.

Keys and Items

- Question: what is the difference between a "key" and an "item"?

Keys and Items

- Question: what is the difference between a "key" and an "item"?
  - An item contains a key, and possibly other pieces of information as well.
  - The key is just the part of the item that we use for sorting/searching.
- For example, the item can be a student record and the key can be a student ID.

Multiple Keys

- In actual applications, we oftentimes want to search or sort by different criteria.
- For example, we may want to search a customer database by ???

Multiple Keys

- In actual applications, we oftentimes want to search or sort by different criteria.
- For example, we may want to search a customer database by:
  - Customer ID.
  - Last name.
  - First name.
  - Phone.
  - Address.
  - …

Multiple Keys

- Accommodating the ability to search by different criteria (i.e., allow multiple keys) is a standard topic in a databases course.
- However, the general idea is fairly simple:
- Define a **primary key**, that is unique.
  - That is why we all have things such as:
  - Social Security number.
  - UTA ID number.
  - Customer ID number, and so on.
- Build a main symbol table based on the primary key.
- For any other field (like address, phone, last name) that we may want to search by, build a separate symbol table, that simply maps values in this field to primary keys.
  - The "key" for each such separate symbol table is NOT the primary key.

Generality of Search by Key

- In this course we will only discuss searching by a single key.
  - This is the problem that is relevant for an algorithms course.
- The previous slides hopefully have convinced you that if you can search by a single key, you can easily accommodate multiple keys as well.
  - That topic is covered in standard database courses.

Overview

- We will now see some standard ways to implement symbol tables.
- Some straightforward ways use arrays and lists.
  – Simple implementations, problematic performance or severe limitations.
- The most commonly used methods use trees.
  – Relatively simple implementations (but more complicated than array/list-based implementations).
  – Good performance.

Key-Indexed Search

- Suppose that:
  - The keys are distinct positive integers, that are sufficiently small.
    - What exactly we mean by "sufficiently small" will be clarified in a bit.
    - "Distinct" means that no two items share the same key.
  - We store our items in an array (so, we have an array-based implementation).
- How would you implement symbol tables in that case?
  - How would you support insertions, deletions, search?

Key-Indexed Search

- Keys are indices into an array.
- Initialization???
- Insertions???
- Deletions???
- Search???

Key-Indexed Search

- Keys are indices into an array.
- Initialization: set all array entries to null, O(N) time.
- Insertions, deletions, search: Constant time.
- Limitations:
  - Keys must be unique.
    - This can be OK for primary keys, but not for keys such as last names, that are not expected to be unique.
  - Keys must be small enough so that the array fits in memory.
- In summary: optimal performance, severe limitations.

Unordered Array Implementation

- Note: this is different than the key-indexed implementation we just talked about.
- Key idea: just throw items into an array.
- Implementation and time complexity:
  - Initialization???
  - Insert?
  - Delete?
  - Search?

Unordered Array Implementation

- Note: this is different than the key-indexed implementation we just talked about.
- Key idea: just throw items into an array.
- Initialization: initialize all entries to null.
  - Linear time.
- Insert: place the new item at the end.
  - Constant time.
- Delete: remove the item, move all subsequent items to fill in the gap.
  - Linear time. This is a problem.
- Search: scan the array, until you find the key you are looking for.
  - Linear time. This is a problem.

Variations

- Unordered list implementation:
  - Linear time for deletion and search.
- Ordered array implementation.
  - Linear time for insertion and deletion.
  - Logarithmic time for search: **binary search** (rings a bell?)
- Ordered list implementation.
  - Linear time for insertion, deletion, search.
- Filling in the details on these variations is left as an exercise.
- However, each of these versions requires linear time for at least one of insertion, deletion, search.
  - We want methods that take **at most logarithmic time** for insertions, deletions, and searches.

Search Trees

- Preliminary note: "search trees" as a term does **NOT** refer to a specific implementation of symbol tables.
  - This is a very common mistake.
- The term refers to a **family of implementations**, that may have **different properties**.
- We will see soon specific implementations with good properties, such as:
  - 2-3-4 trees.
  - Red-black trees.

Search Trees

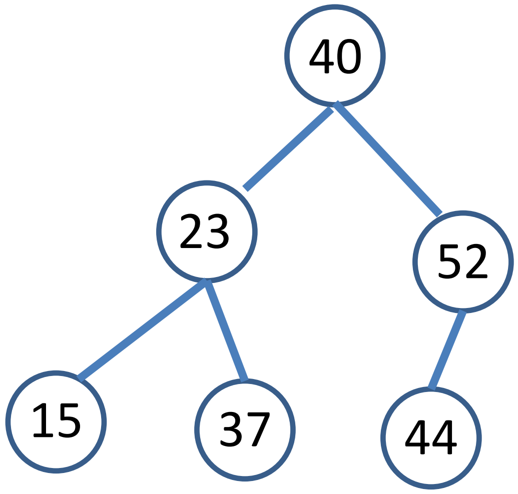
- What all search trees have in common is the implementation of search.
- Insertions and deletions can differ, and have important implications on overall performance.
- The main goal is to have insertions and deletions that:
  - Are efficient (at most logarithmic time).
  - Leave the tree balanced, to support efficient search (at most logarithmic time).

Binary Search Trees

- Definition: a binary search tree is a binary tree where:
- Each internal node contains an item.
  - External nodes (leaves) do not contain items.
- The item at each node is:
  - Greater than or equal to all items on the left subtree.
  - Less than all items in the right subtree.

Binary Search Trees

- Parenthesis: is this a binary tree?
- According to the definition in the book (that we use in this course), no, because one node has only one child.
- However, a binary tree can only have an odd number of nodes.
- What are we supposed to do if the number of items is even?
  - Hint: look back to the previous definition.



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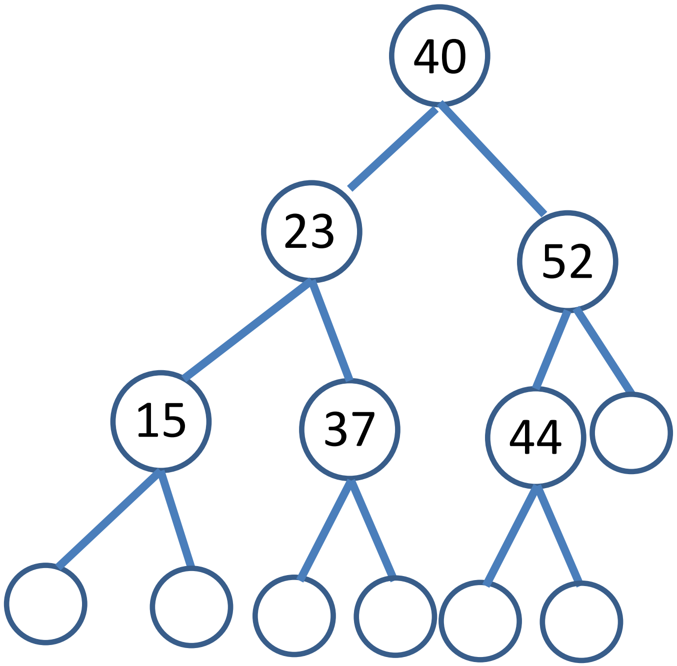
- However, a binary tree can only have an odd number of nodes.
- What are we supposed to do if the number of items is even?
- We make the convention that items are only stored at internal nodes.
- Leaves exist, but they do not contain items.
- To simplify, we will **not** be showing leaves.



Binary Search Trees

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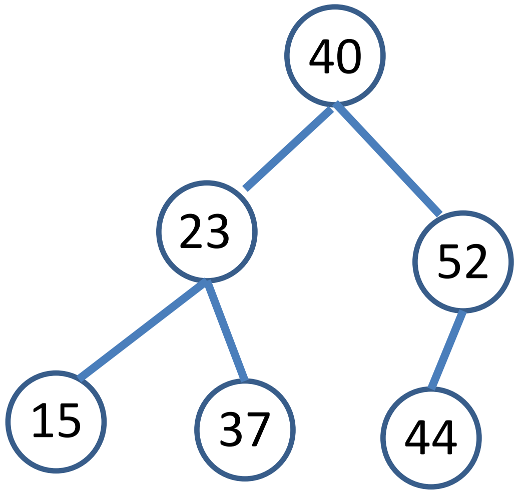
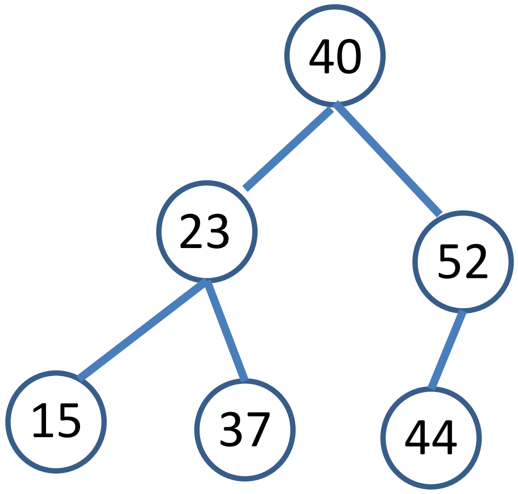


Binary Search Trees

- So, is this a binary tree?
- We will make the convention that yes, this is a binary tree whose leaves contain no items and are not shown.



Binary Search Trees

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  - Less than all items in the right subtree.

- How do we implement search?

Binary Search Trees

- Definition: a binary search tree is a binary tree where the item at each node is:
  - Greater than or equal to all items on the left subtree.
  - Less than all items in the right subtree.

- search(tree, key)
  - if (tree == null) return null
  - else if (key == tree.item.key)
    - return tree.item
  - else if (key < tree.item.key)
    - return search(tree.left_child, key)
  - else return search(tree.right_child, key)

Performance of Search

- Note: so far we have said nothing about how to implement insertions and deletions.
- Given that, what can we say about the worst-case time complexity of search?

Performance of Search

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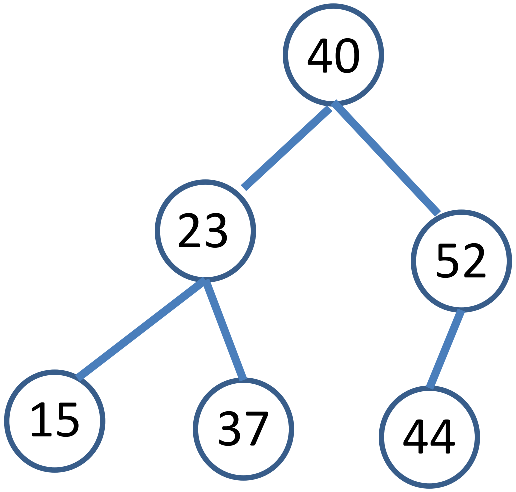
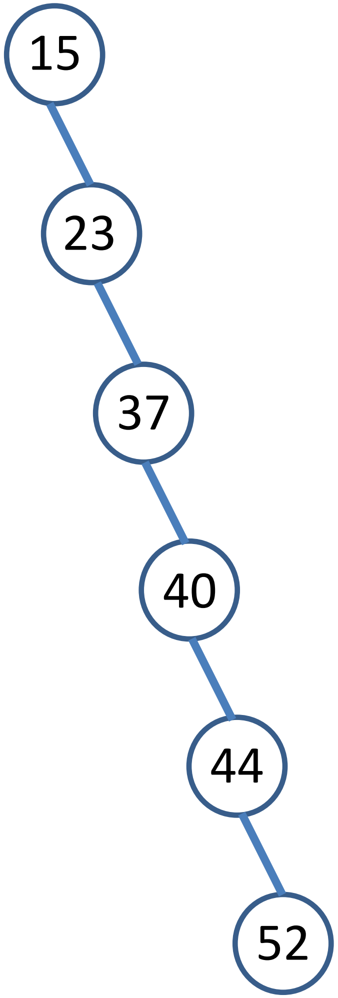
- Given that, what can we say about the worst-case time complexity of search?
- A binary tree can be perfectly balanced or maximally unbalanced.





Performance of Search

- Note: so far we have said nothing about how to implement insertions and deletions.
- Given that, what can we say about the worst-case time complexity of search?
- Search takes time that is in the worst case linear to the number of items.
  - This is not very good.
- Search takes time that is linear to the height of the tree.
- For balanced trees, search takes time logarithmic to the number of items.
  - This is good.
- So, the challenge is to make sure that insertions and deletions leave the tree balanced.

Naïve Insertion

- To insert an item, the simplest approach is to go down the tree until finding a leaf position where it is appropriate to insert the item.
- Pseudocode ???

Naïve Insertion

- To insert an item, the simplest approach is to go down the tree until finding a leaf position where it is appropriate to insert the item.
- insert(tree, item)
  - if (tree == null) return new tree(item.key)
  - else if (item.key < tree.item.key)
    - tree.left_child = insert(tree.left_child, item)
  - else if (item.key > tree.item.key)
    - tree.right_child = insert(tree.right_child, item)
  - return tree

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Why do we use line
tree.left_child = insert(tree.left_child, item)
instead of line
insert(tree.left_child, item)

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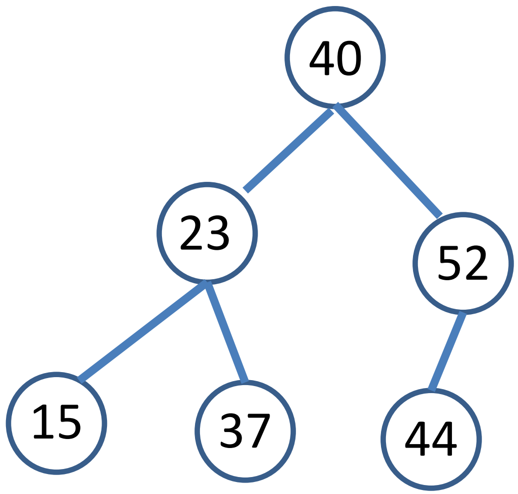
  - return tree

Answer: To handle the base case, where we return a new node, and the parent must make this new node a child.

Naïve Insertion

- Inserting a 39:



Naïve Insertion

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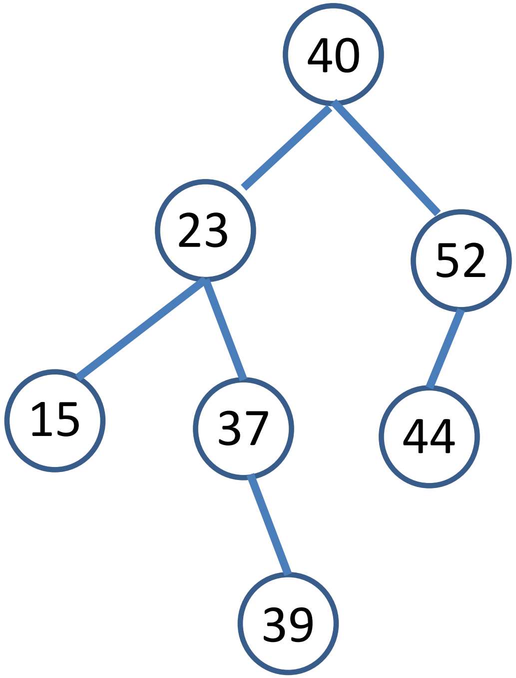


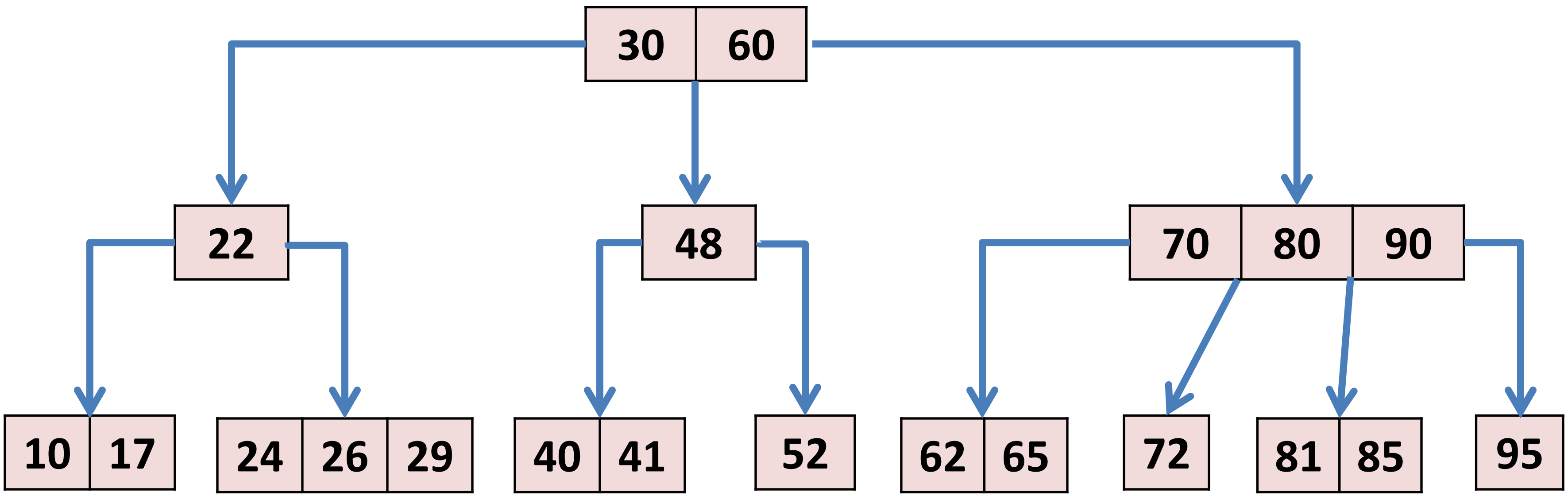
Naïve Insertion

- If items are inserted in random order, the resulting trees are reasonably balanced.
- If items are inserted in ascending order, the resulting tree is maximally imbalanced.
- We will next see more sophisticated methods, that guarantee that the resulting tree is balanced regardless of the order of insertions/deletions.

2-3-4 Trees

- A 2-3-4 tree is a tree that either is empty or contains three types of nodes:
- 2-nodes, which contain:
  - An item with key K.
  - A left subtree with keys <= K.
  - A right subtree with keys > K.
- 3-nodes, which contain:
  - Two items with keys K1 and K2, K1 <= K2.
  - A left subtree with keys <= K1.
  - A middle subtree with K1 < keys <= K2.
  - A right subtree with keys > K2.
- 4-nodes, which contain:
  - Three items with keys K1, K2, K3, K1 <= K2 <= K3.
  - A left subtree with keys <= K1.
  - A middle-left subtree with K1 < keys <= K2.
  - A middle-right subtree with K2 < keys <= K3.
  - A right subtree with keys > K3.
- For a 2-3-4 search tree to be called **balanced**, all leaves must be at the **same distance from the root**.
- We will only consider balanced 2-3-4 trees.

Example of 2-3-4 Tree



Search in 2-3-4 Trees

- Search in 2-3-4 trees is a generalization of search in binary search trees.
- For simplicity, **we assume that all keys are unique**.
- Given a search key, at each node select one of the subtrees by comparing the search key with the 1, 2, or 3 keys that are present at the node.
- The time is linear to the height of the tree.
- Since we assume that 2-3-4 trees are balanced, search time is logarithmic to the number of items.
- Question to tackle next:
  - how to implement insertions and deletions so as to guarantee that, when we start with a balanced 2-3-4 tree, the tree remains balanced after the insertion or deletion.

Insertion in 2-3-4 Trees

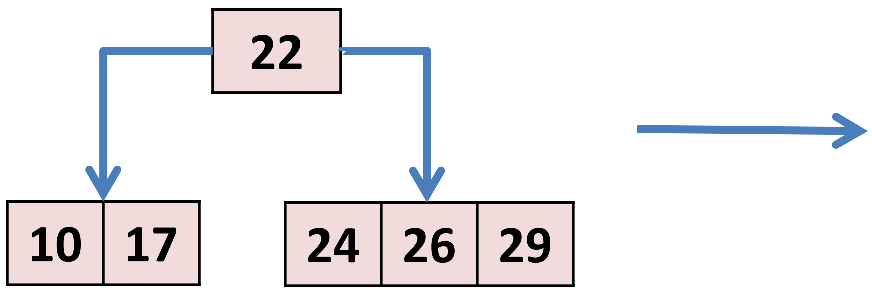
- We follow the same path as if we are searching for the item.
- A simple approach would be to just insert the item at the end of that path.
- However, if we insert the item at a new node at the end, the tree is not balanced any more.
- We need to make sure that the tree remains balanced, so we follow a more complicated approach.

Insertion in 2-3-4 Trees

- Given our key K: we follow the same path as in search.
- On the way:
  - If we find a 2-node being parent to a 4-node, we transform the pair into a 3-node connected to two 2-nodes.
  - If we find a 3-node being parent to a 4-node, we transform the pair into a 4-node connected to two 2-nodes.
  - If the root becomes a 4-node, split it into three 2-nodes.
- These transformations:
  - Are local (they only affect the nodes in question).
  - Do not affect the overall height or balance of the tree (except for splitting a 4-node at the root).
- This way, when we get to the bottom of the tree, we know that the node we arrived at is not a 4-node, and thus it has room to insert the new item.

Transformation Examples

- If we find a 2-node being parent to a 4-node, we transform the pair into a 3-node connected to two 2-nodes, by pushing up the middle key of the 4-node.



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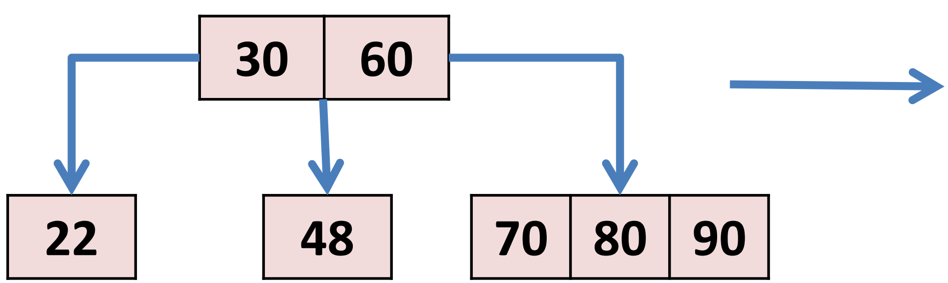

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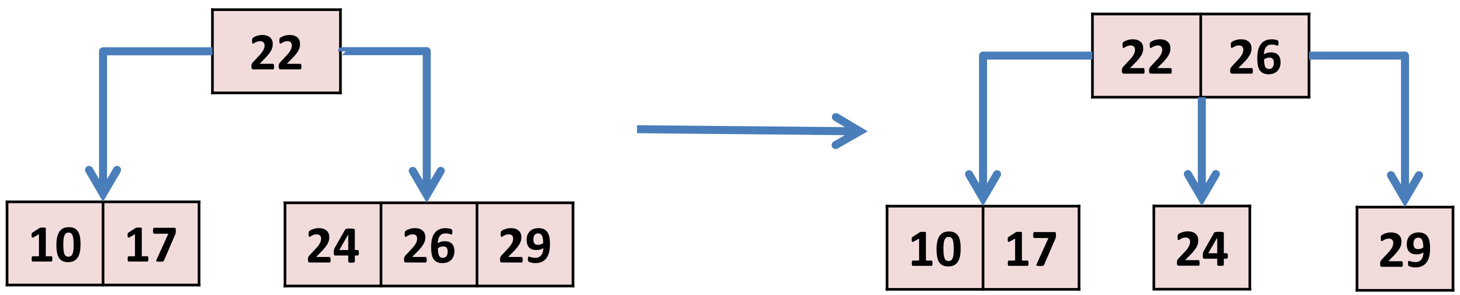

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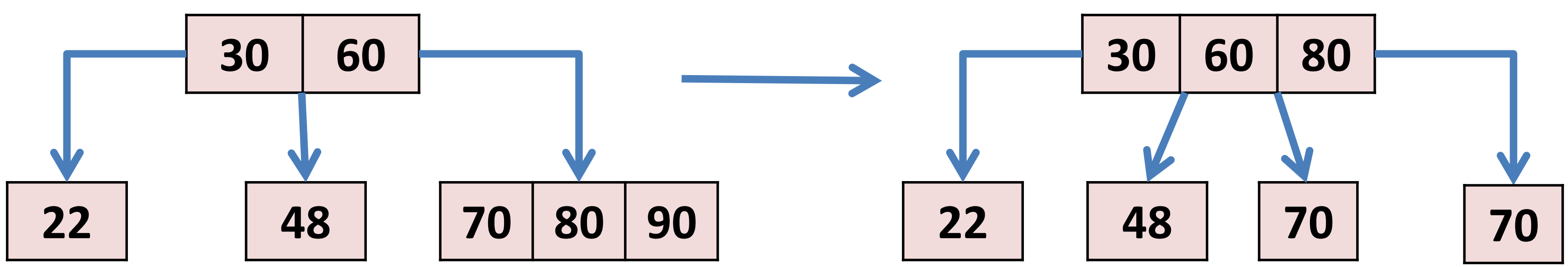
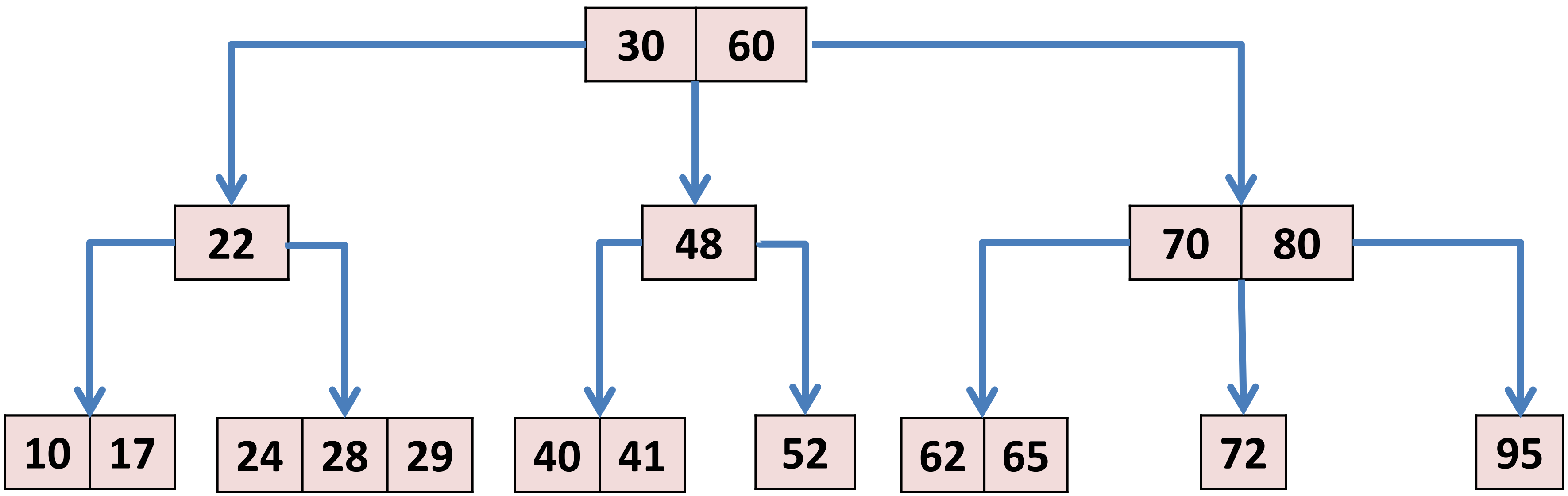


Insertion Example

- Inserting an item with key 25:



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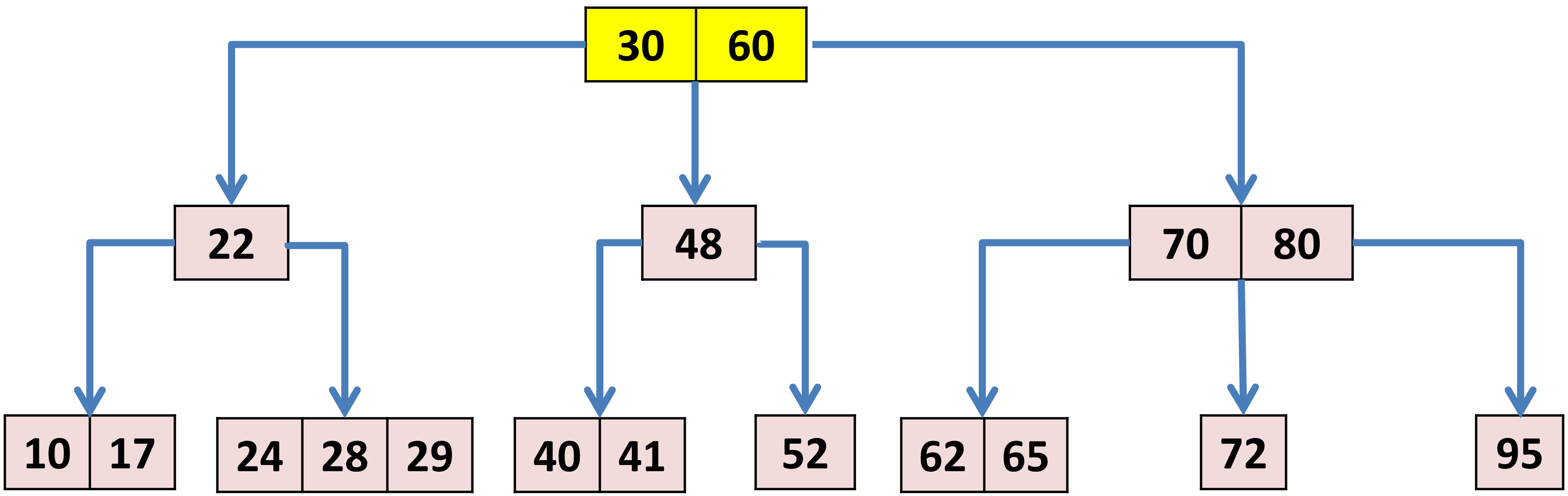

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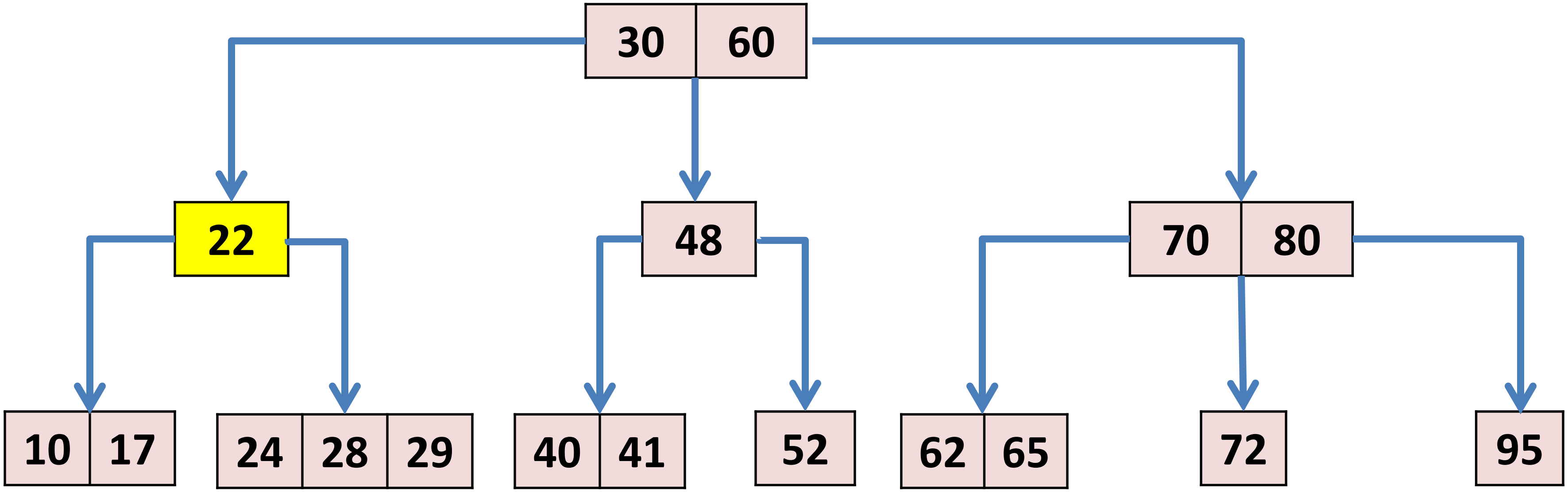

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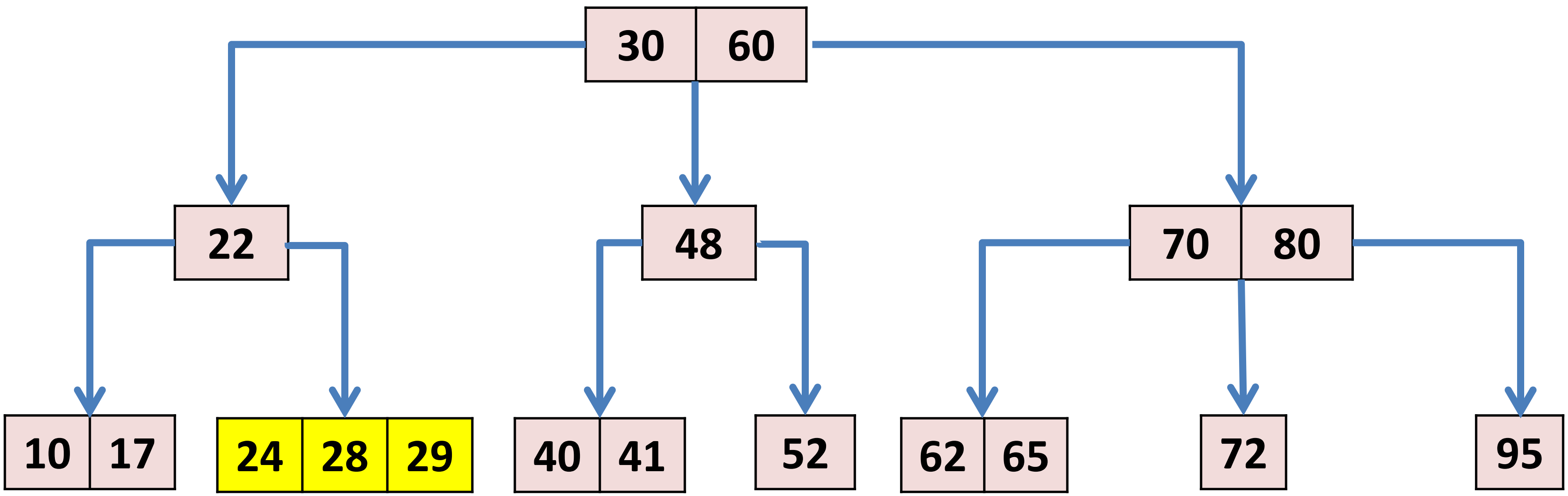

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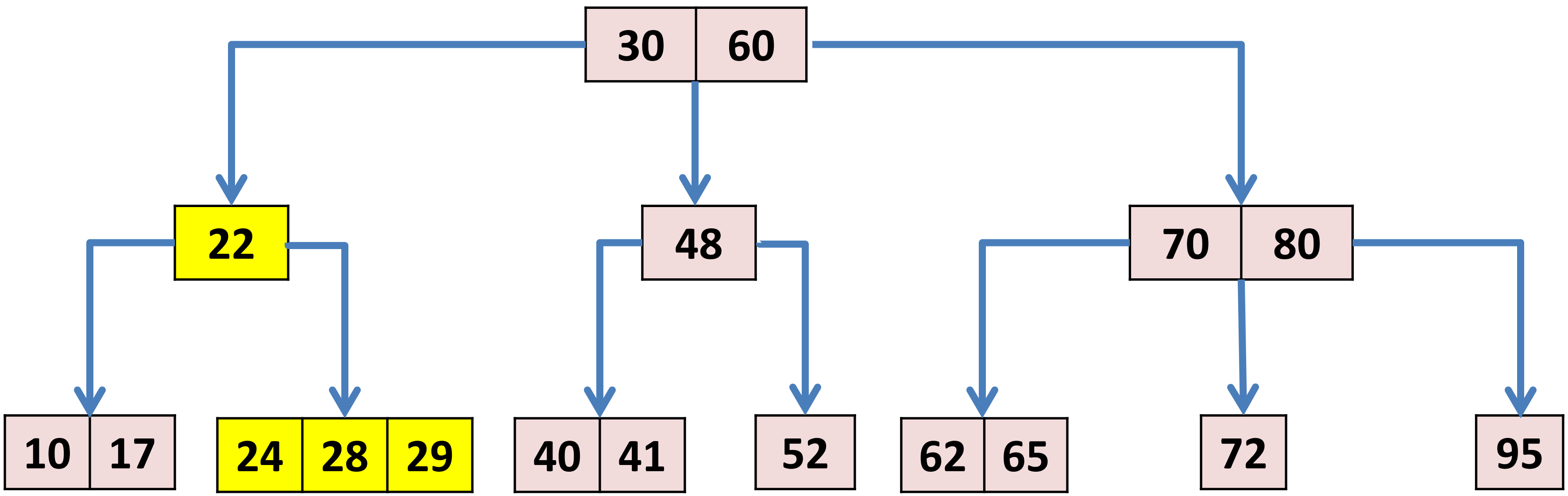

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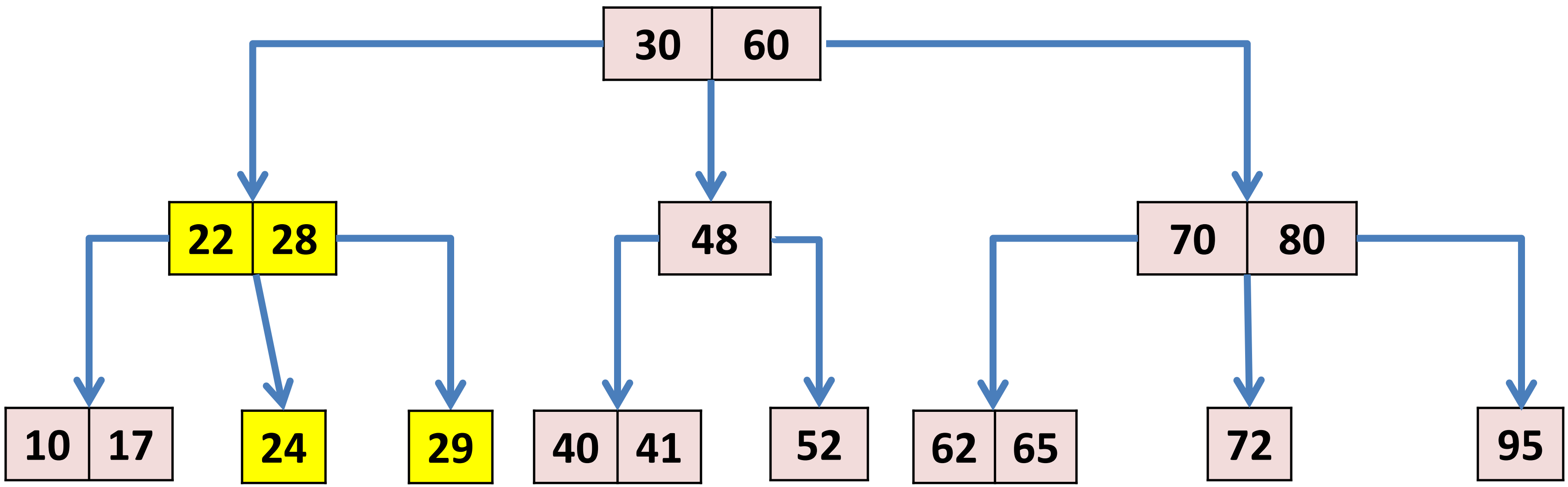
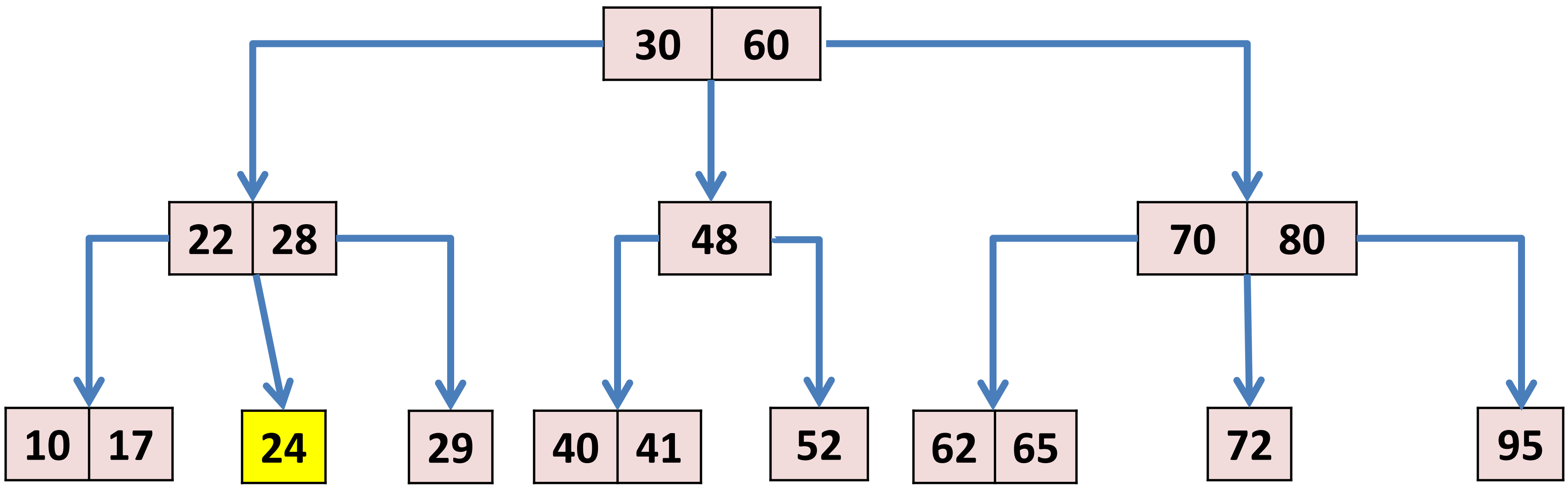


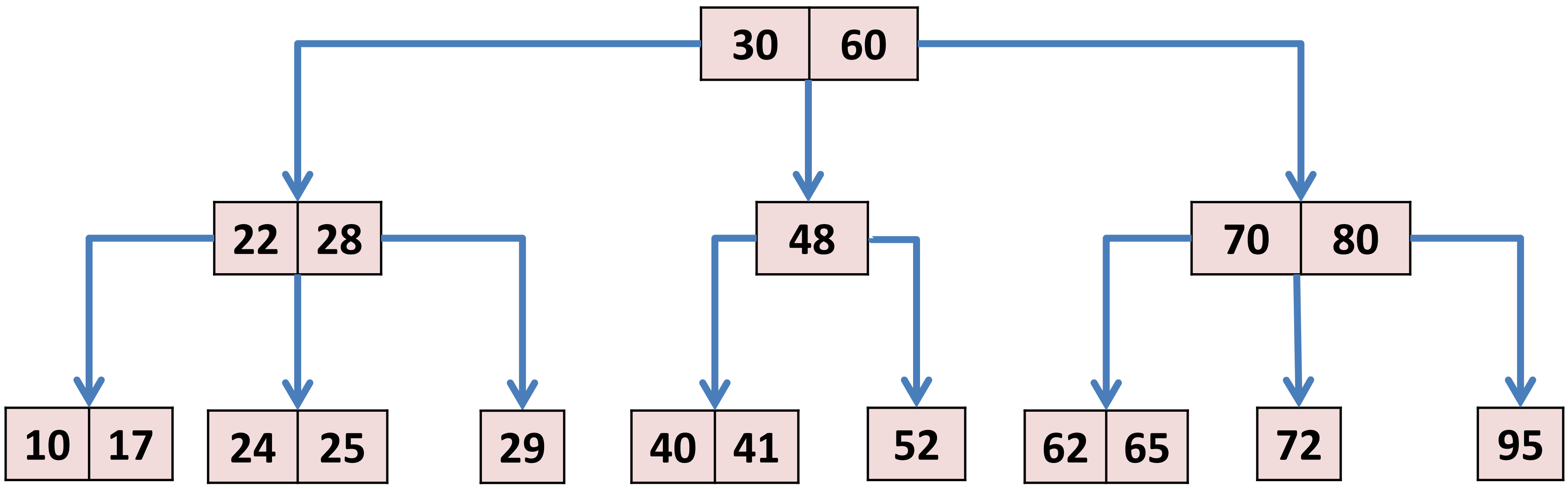
Insertion Example

- Reached the bottom. Make insertion of item with key 25.



Insertion Example

- Next: insert an item with key = 27.



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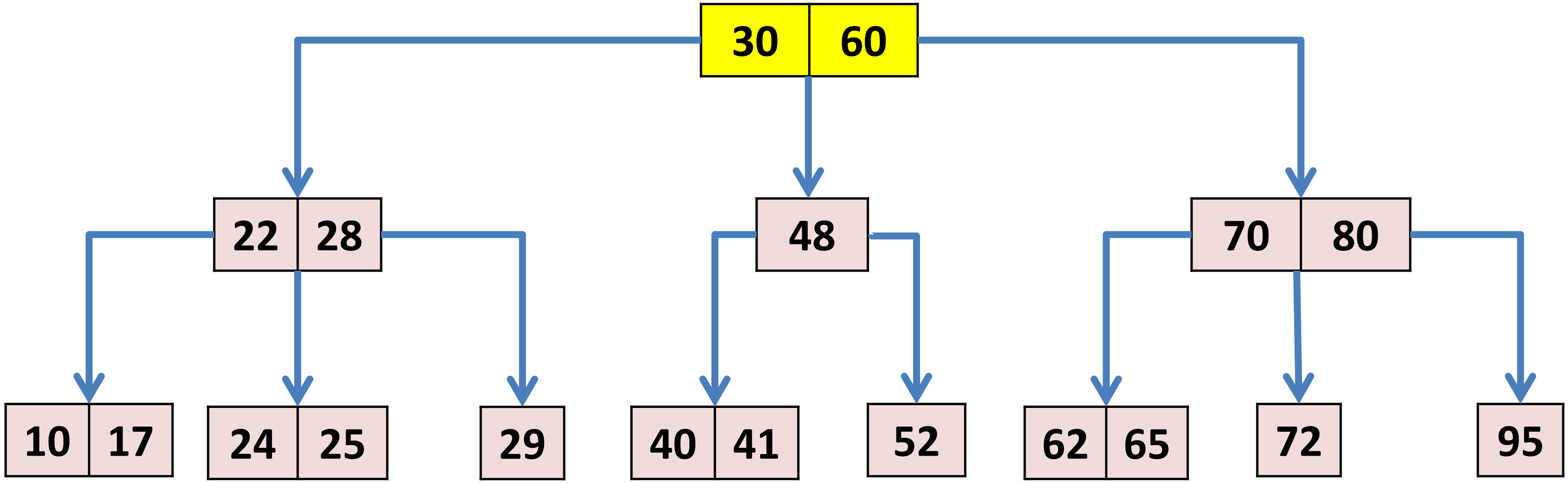

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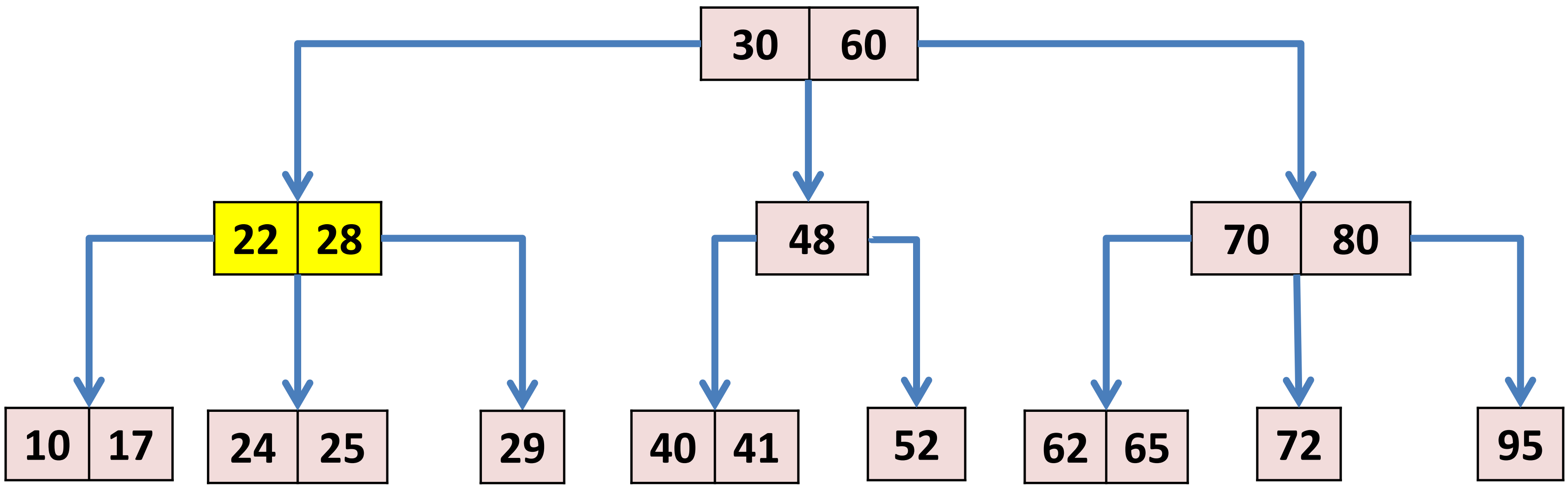

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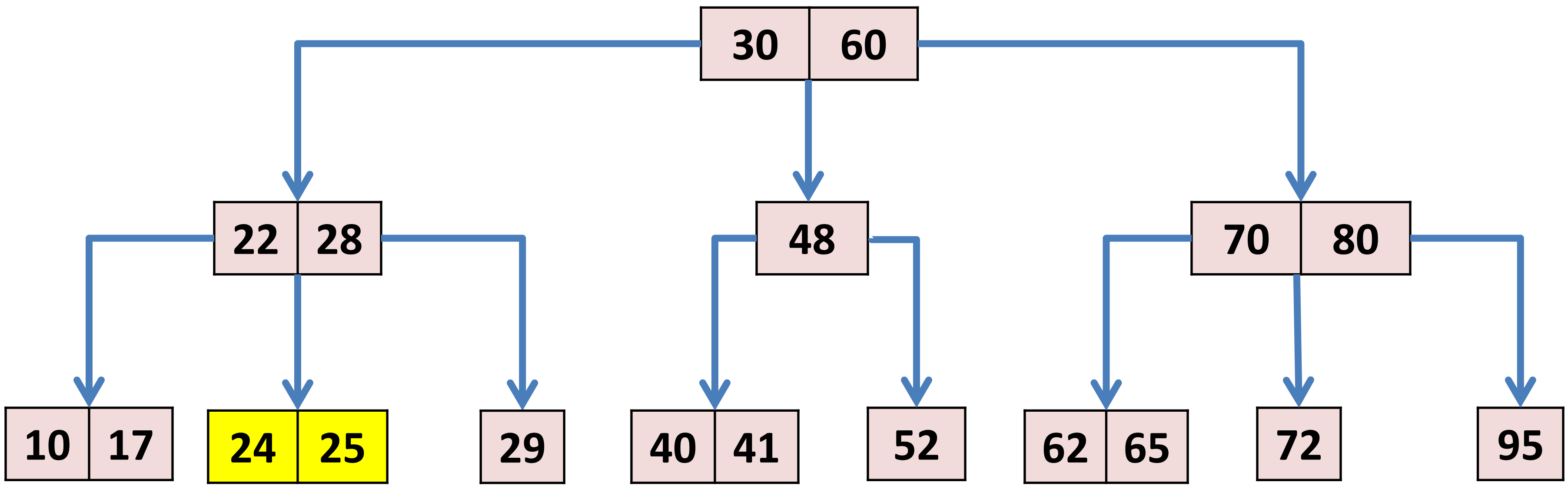

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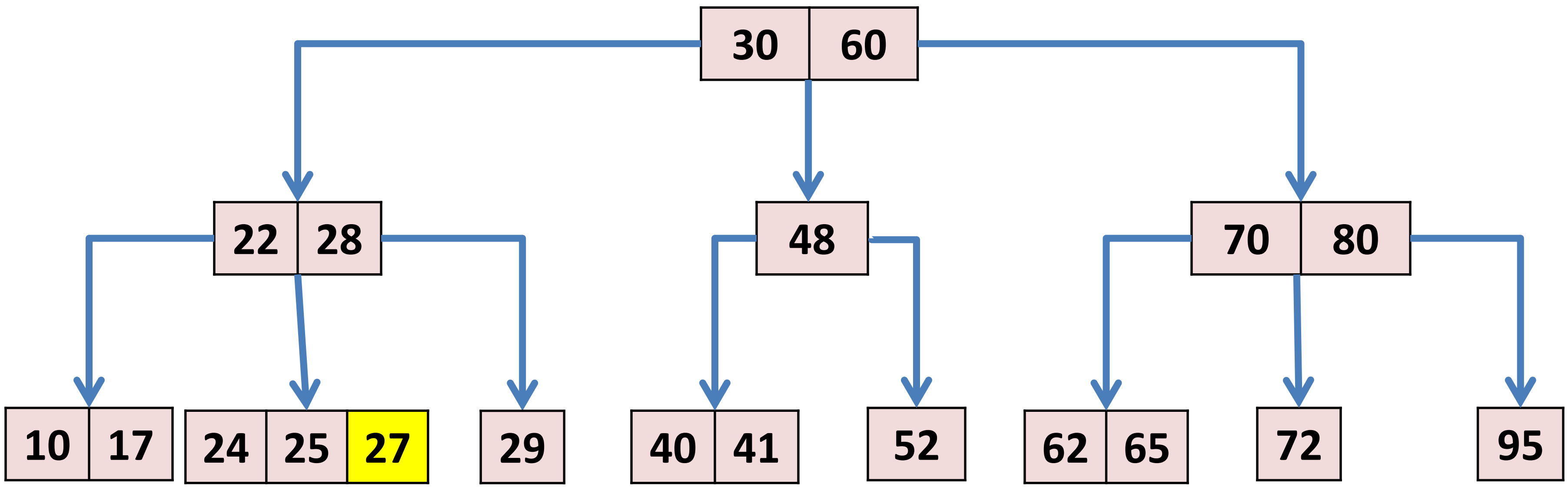

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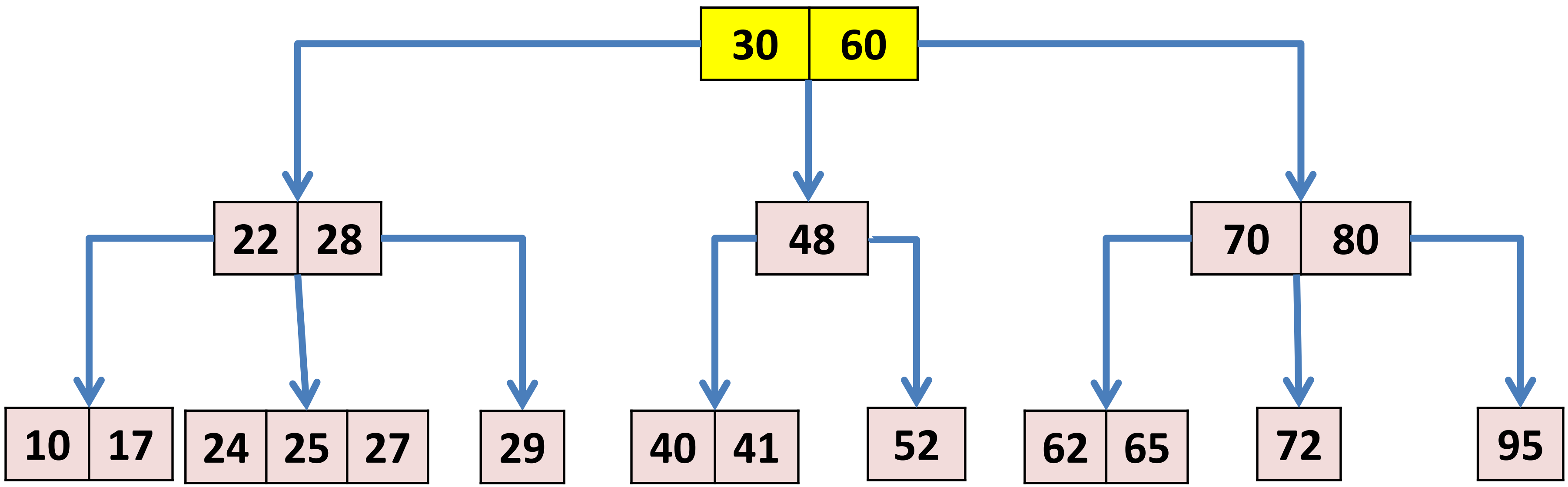

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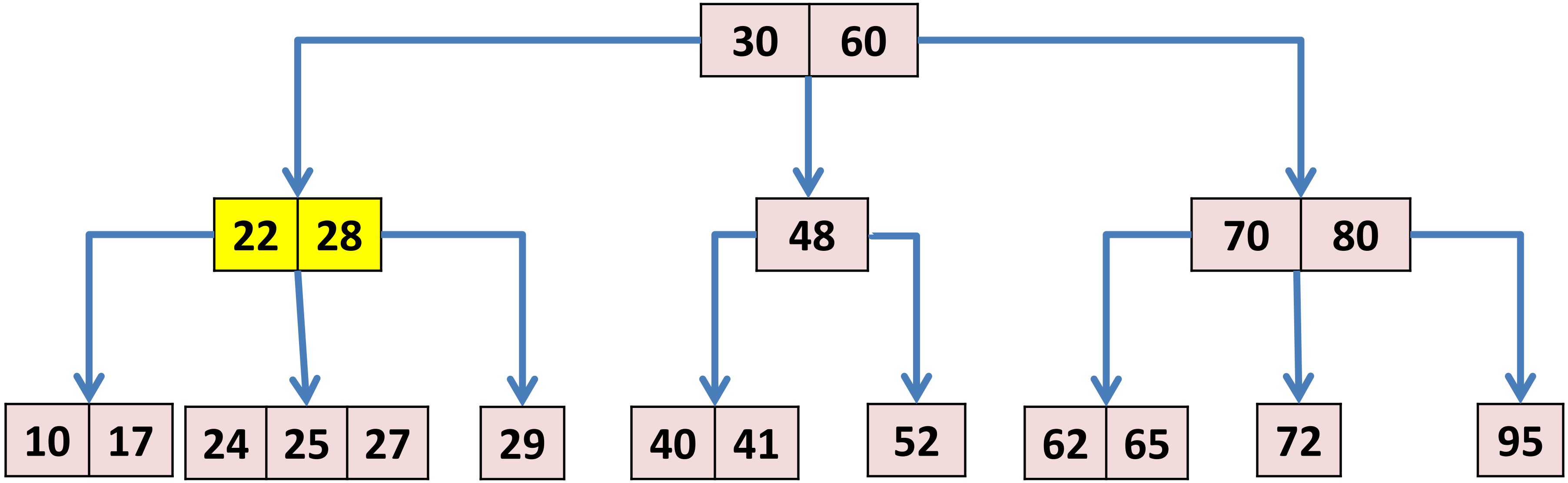

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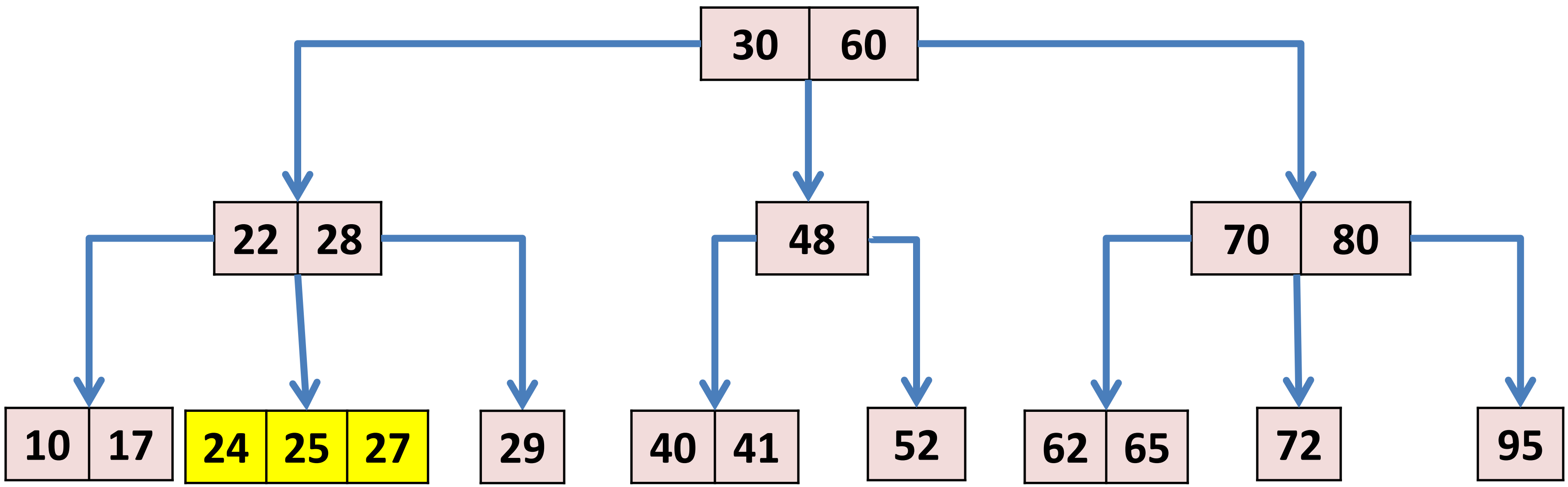

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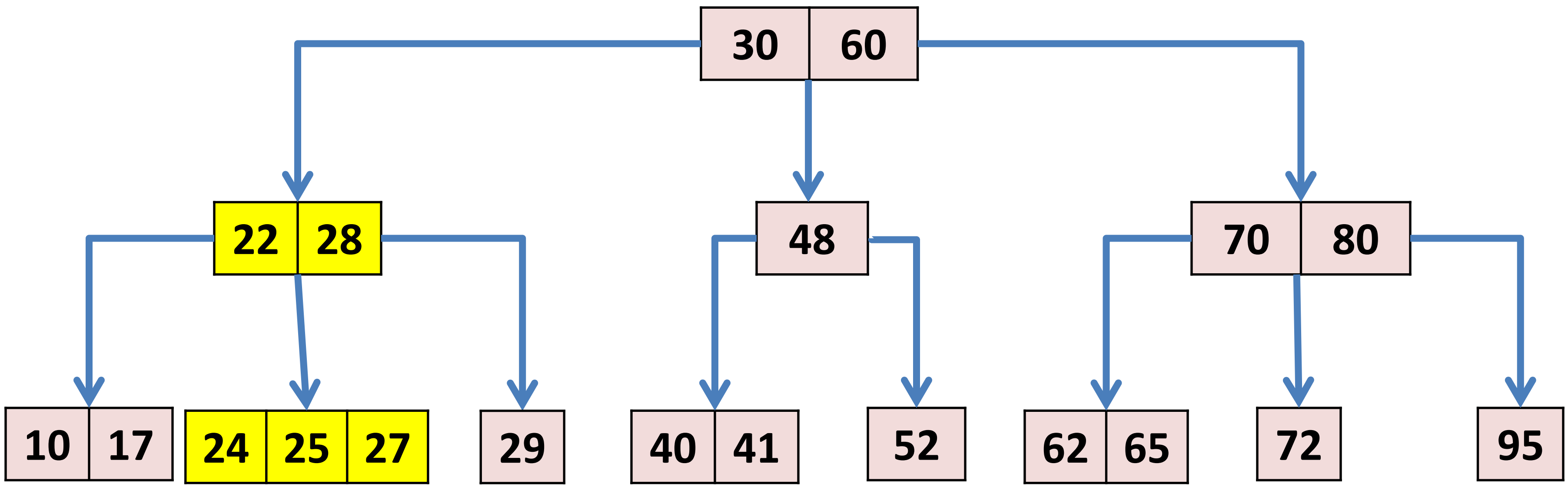

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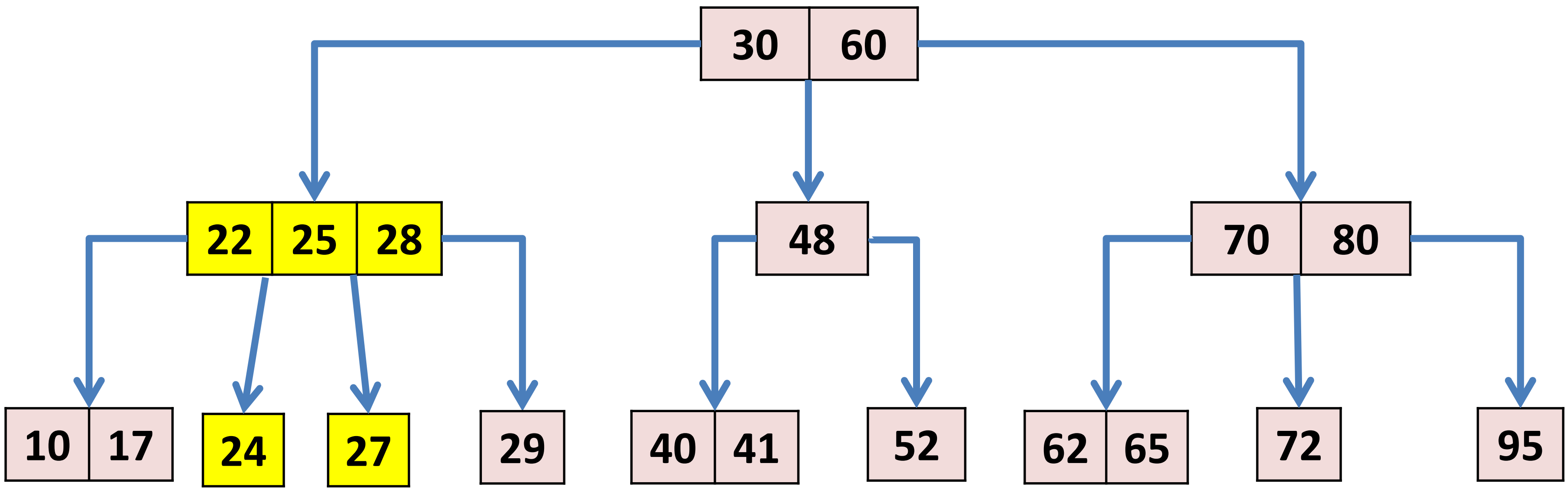
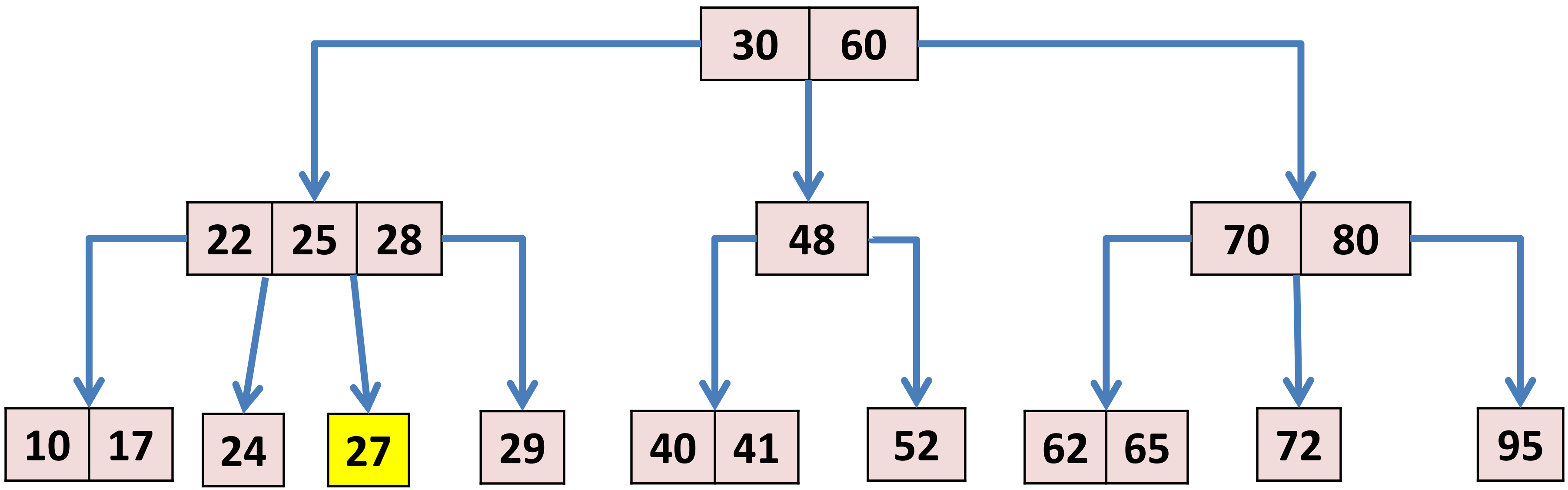


Insertion Example

- Reached the bottom. Make insertion of item with key 26.



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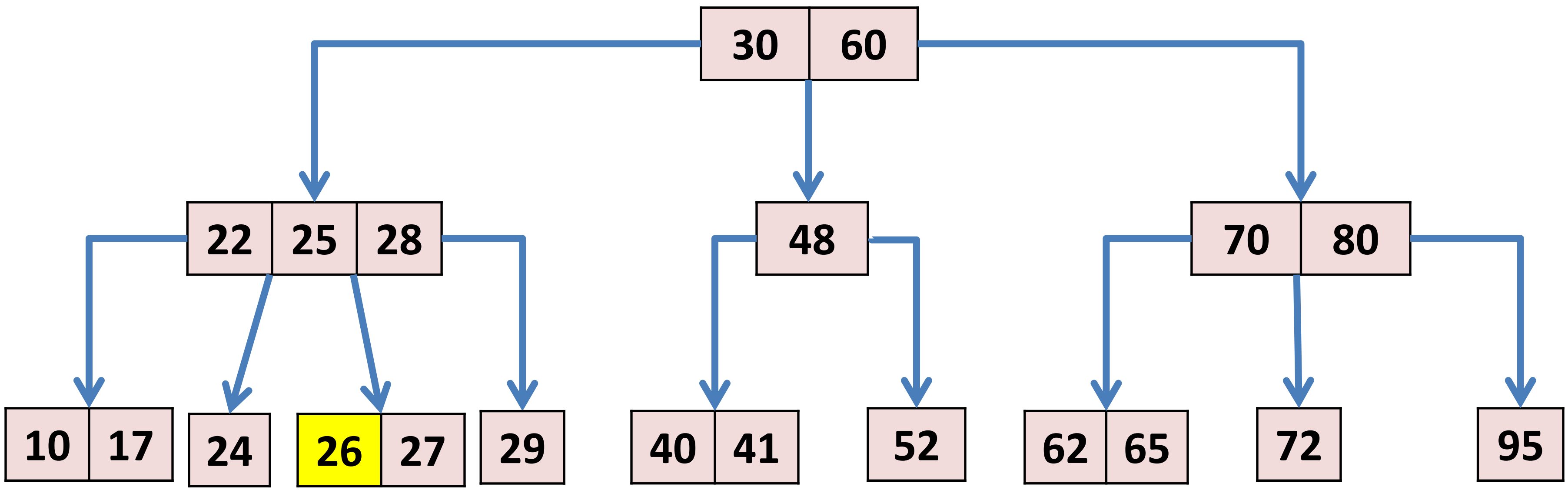

Insertion Example

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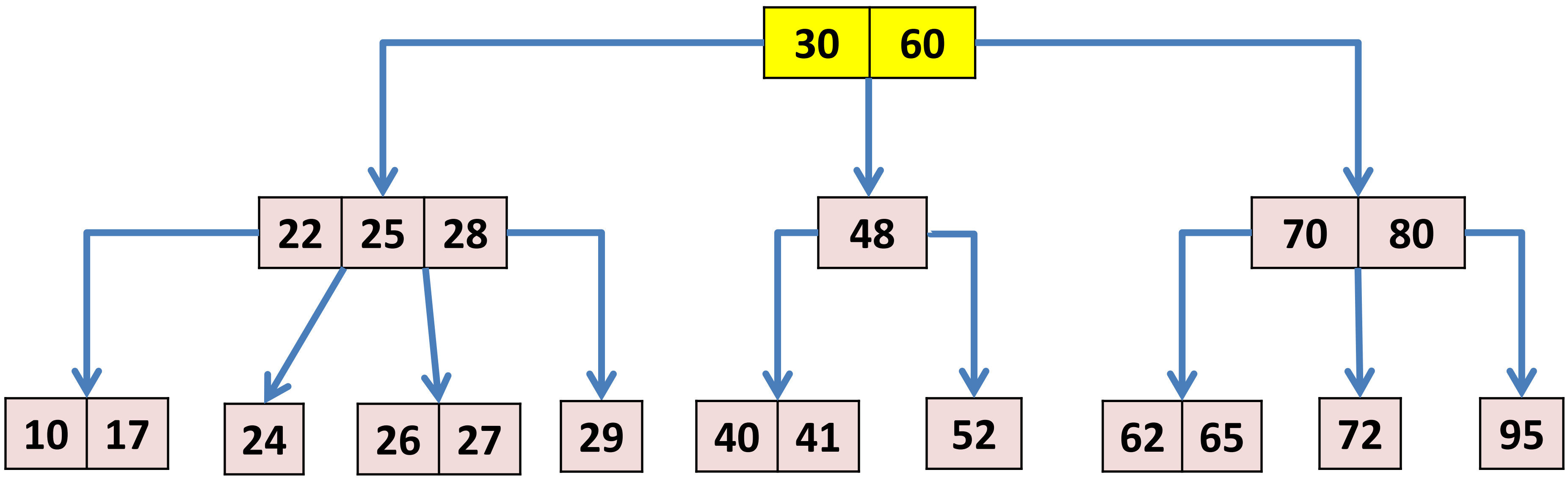

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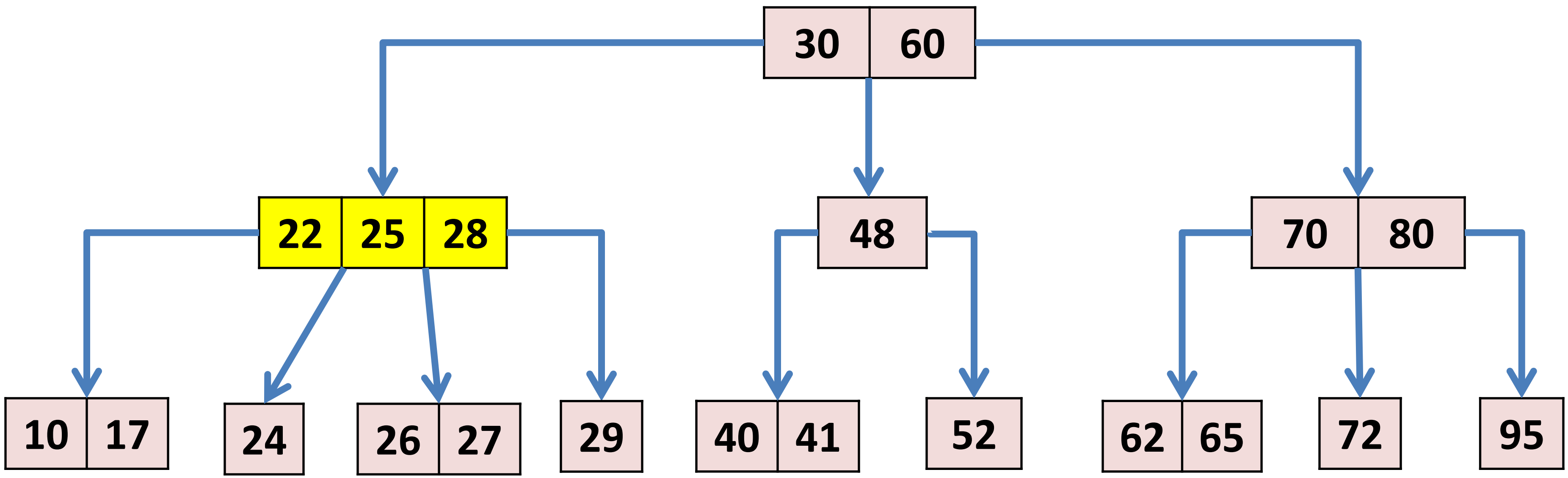
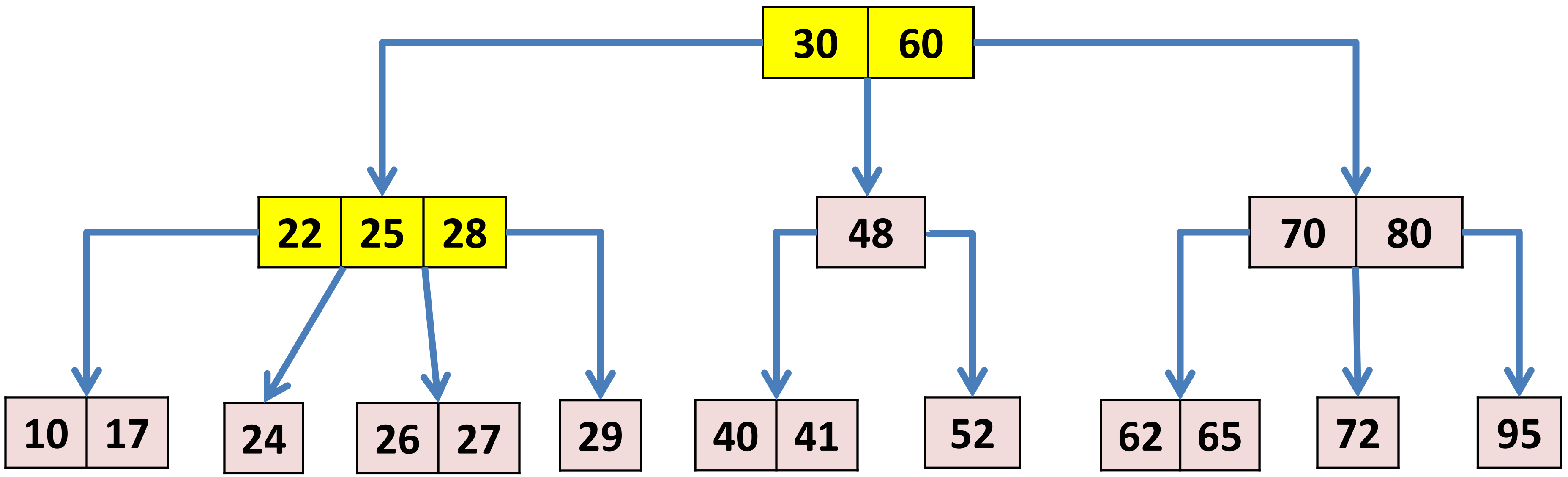


Insertion Example

- Found a 3-node being parent to a 4-node, we must transform the pair into a 4-node connected to two 2-nodes.



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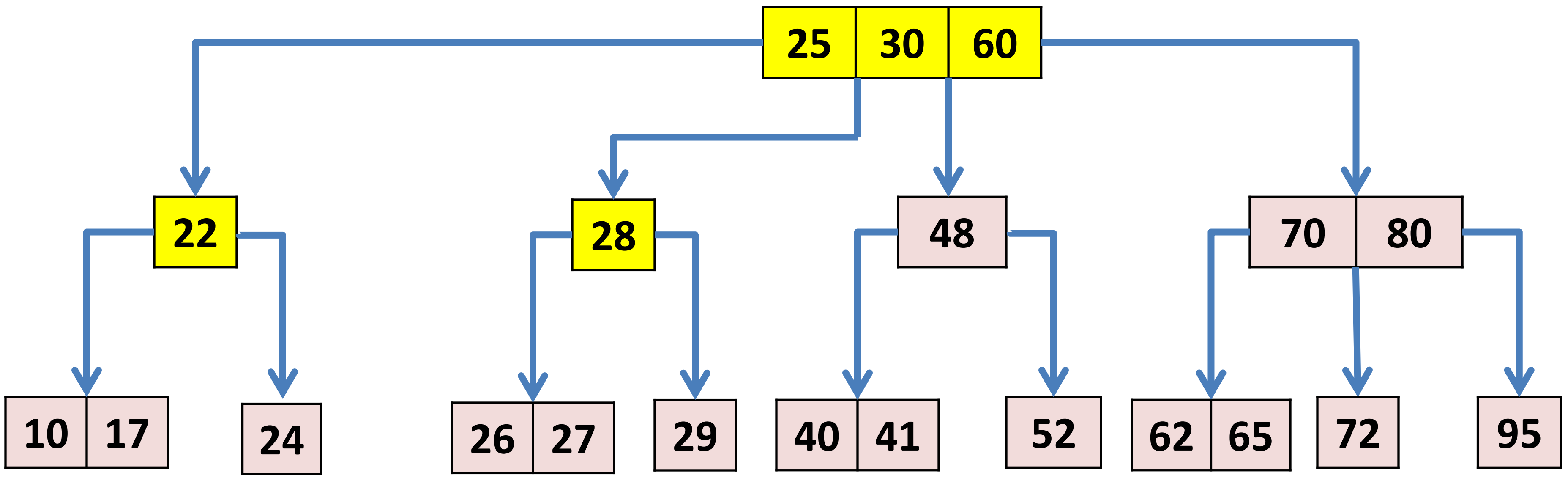
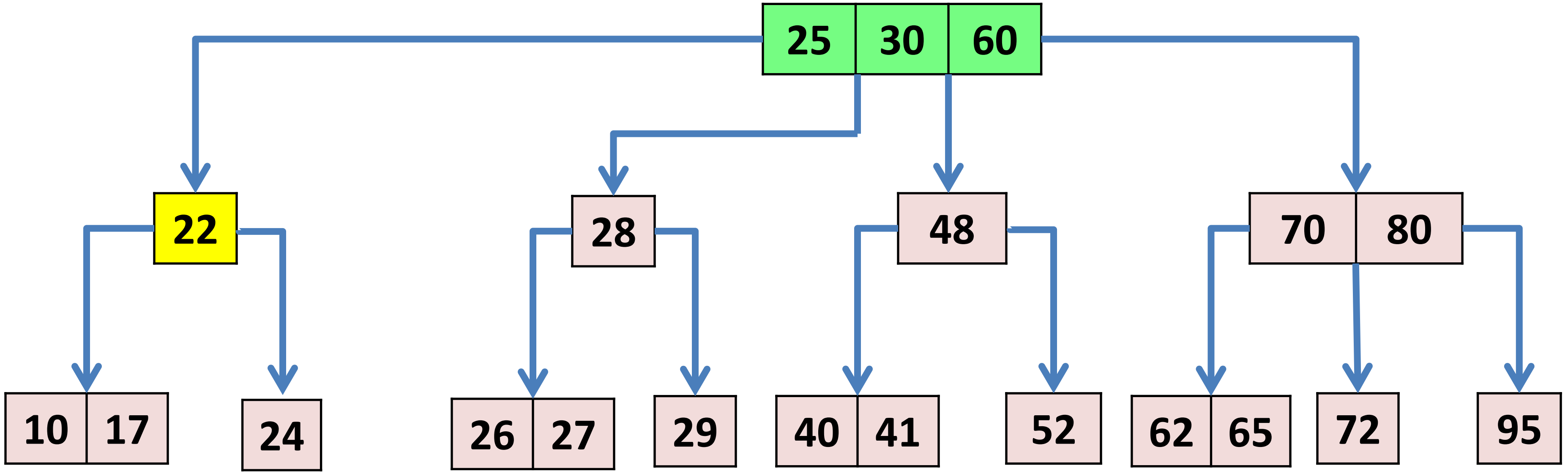


Insertion Example

- Root is 4-node, must split.



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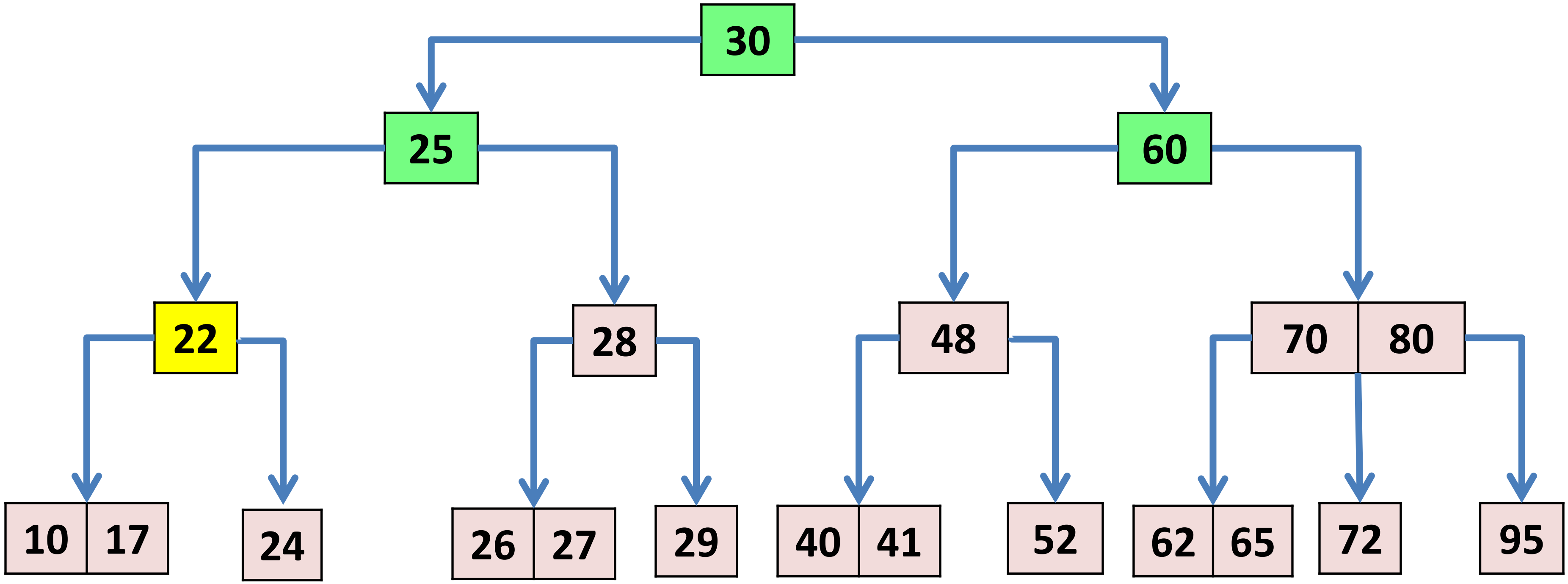
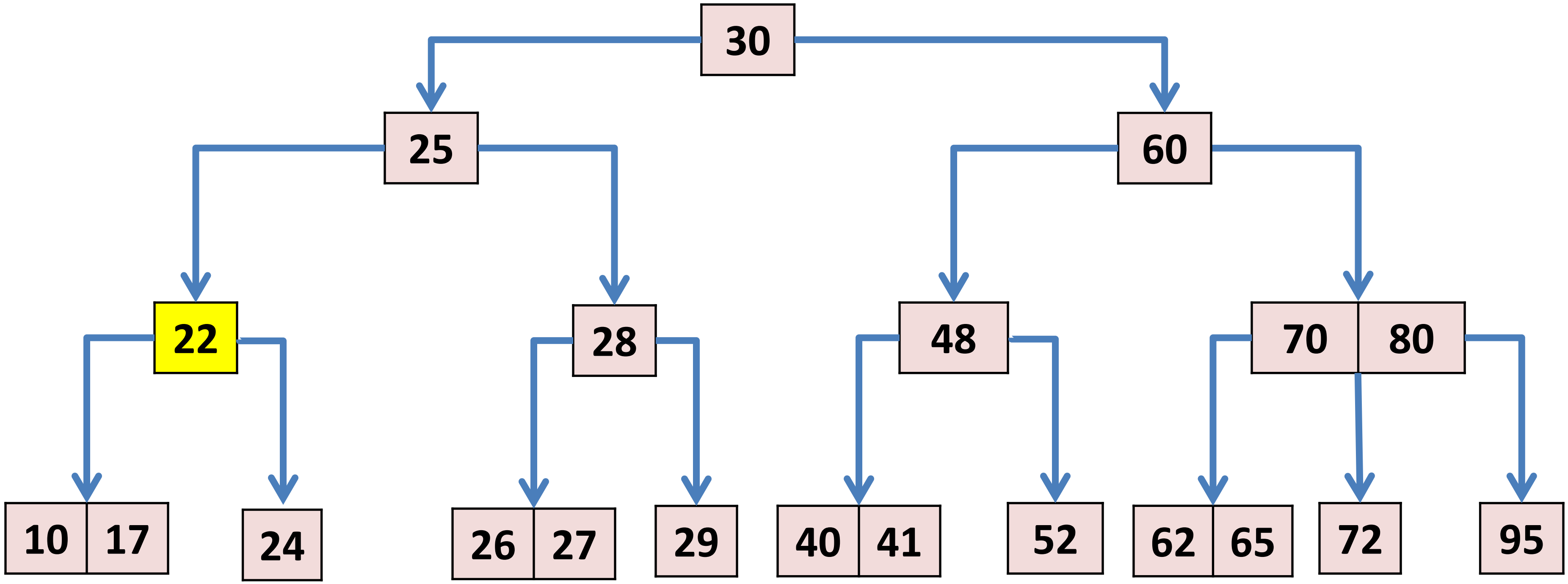


Insertion Example

- Continue with inserting an item with key = 13.



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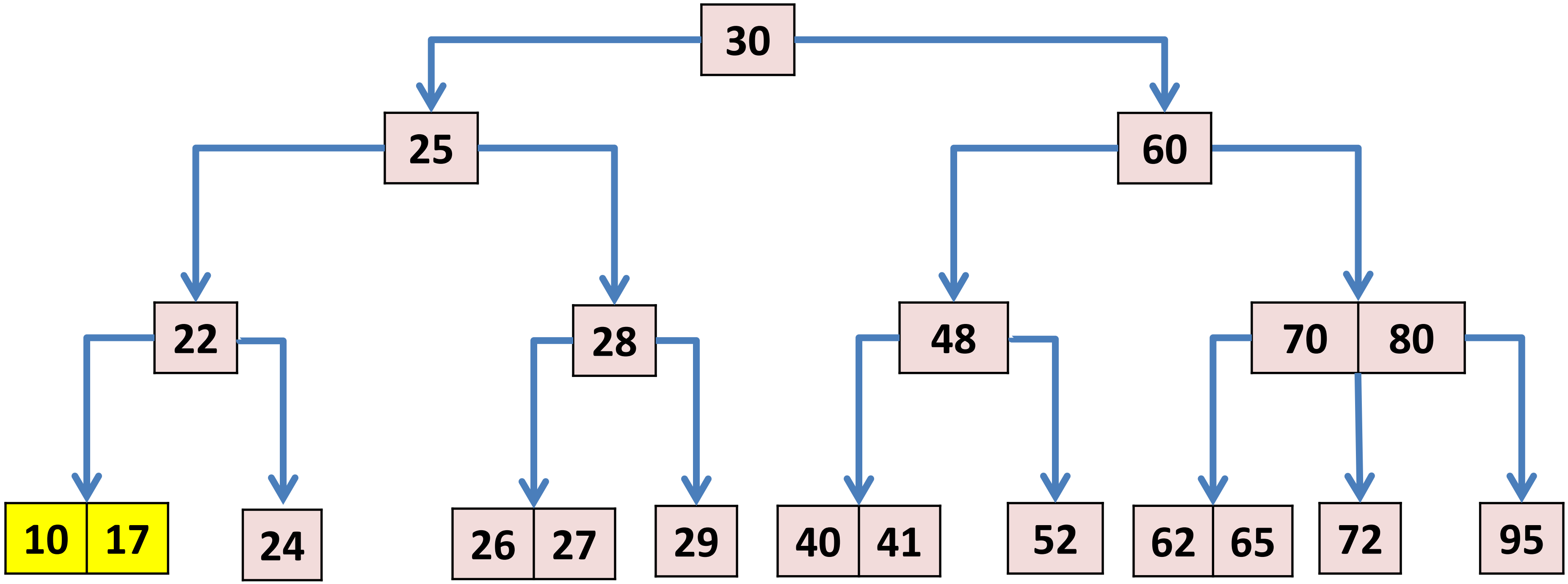

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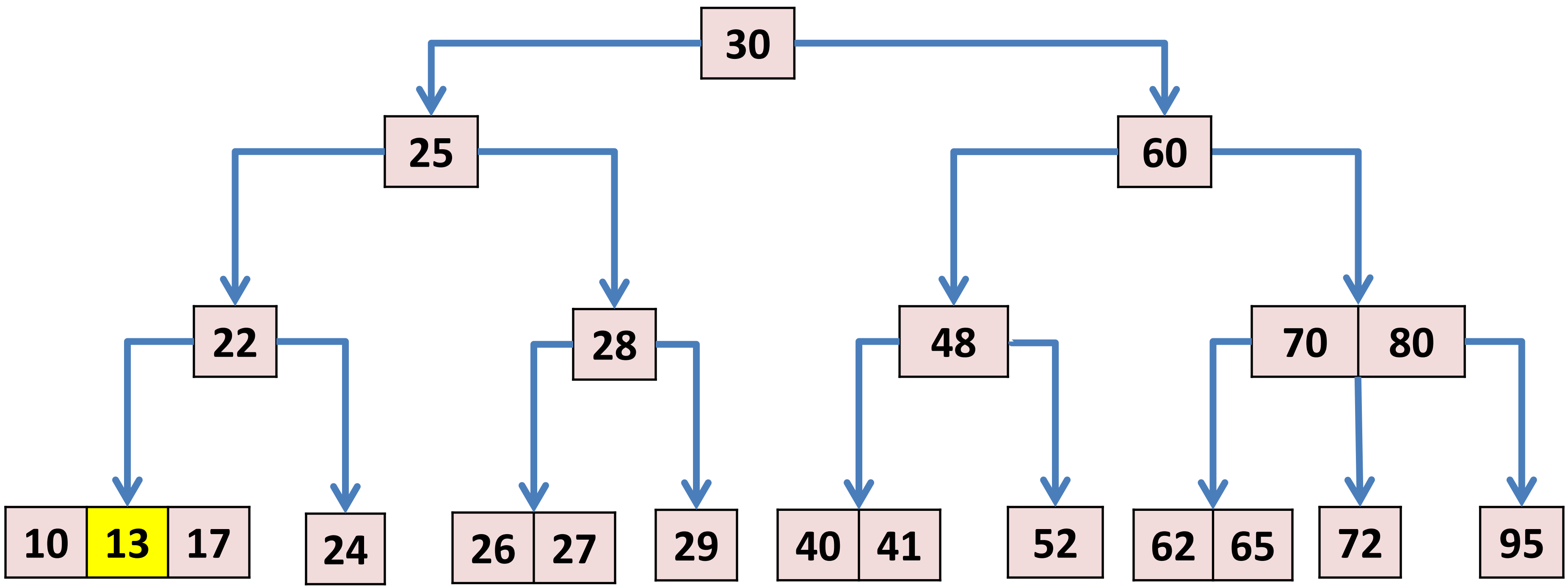

Deletion on 2-3-4 Trees

- More complicated.
- The book does not cover it.
- We will not cover it.
- If you care, you can look it up on Wikipedia.

Red-Black Trees

- Red black trees are an alternative way to view/implement 2-3-4 trees.
- Red links: bind together 3-nodes and 4-nodes into small binary trees.
- Black links: bind the tree together.