Minimum Spanning Trees

CSE 2320 – Algorithms and Data Structures
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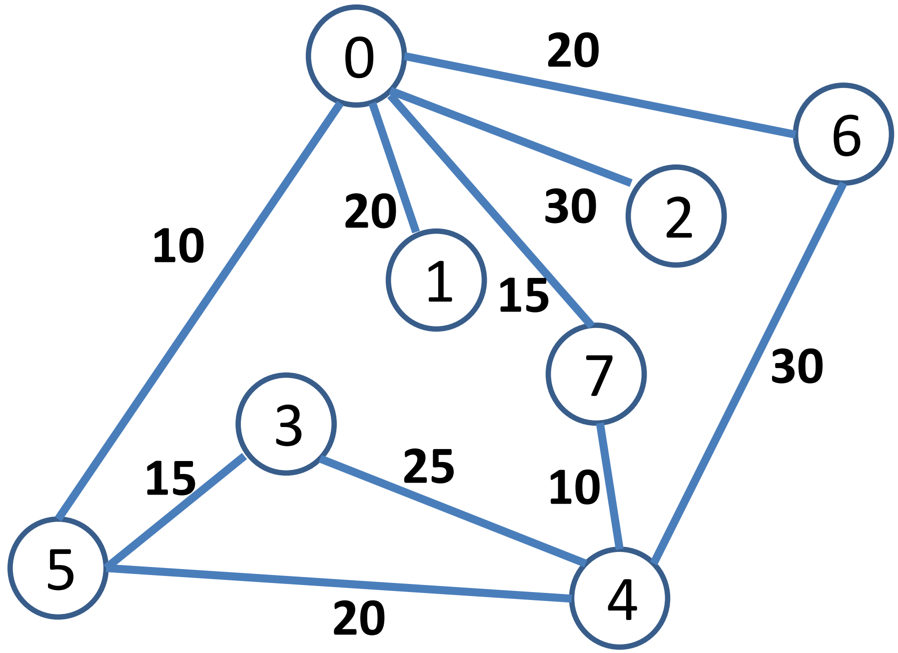
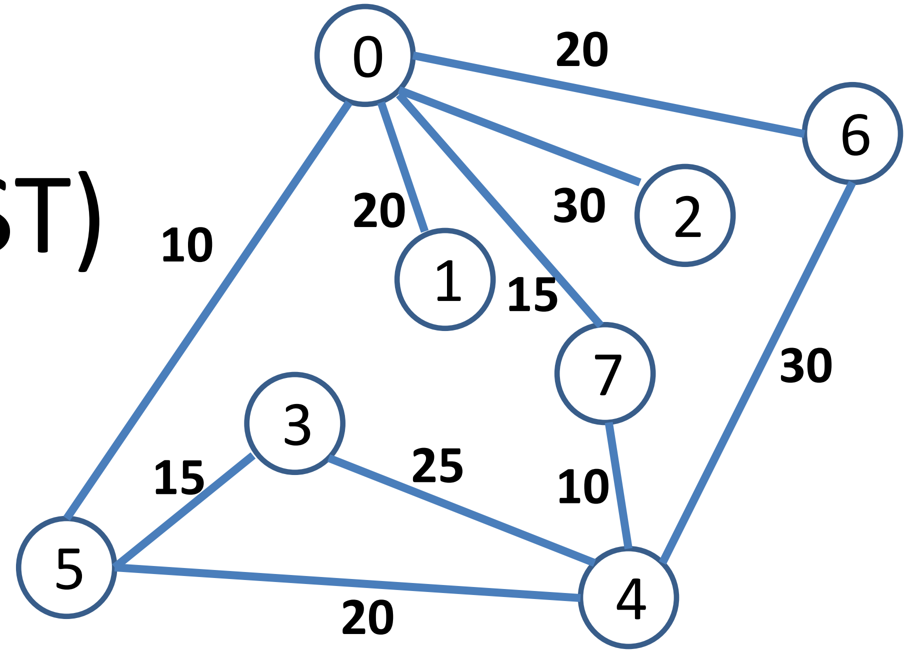
Weighted Graphs

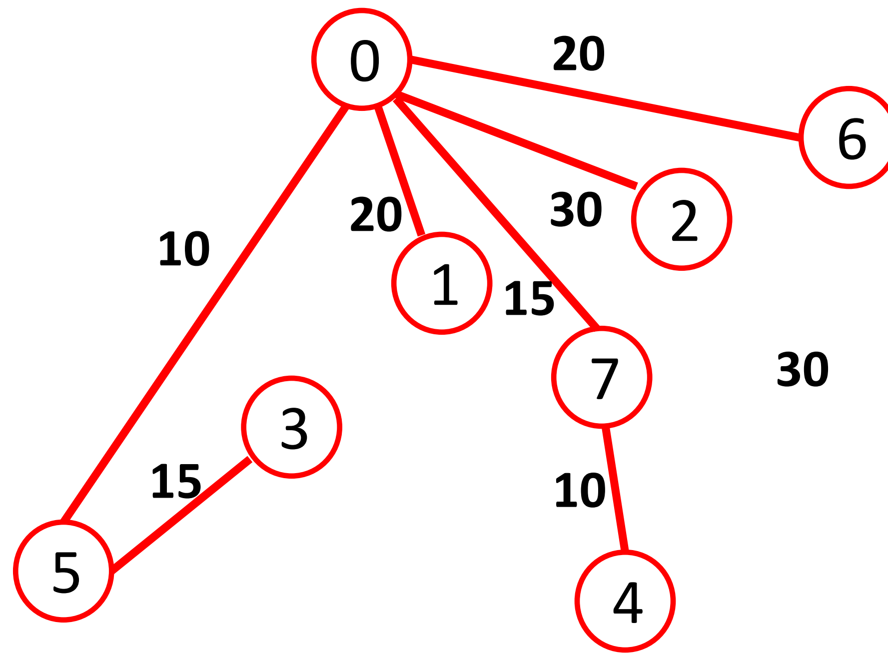
- Each edge has a weight.
- Example: a transportation network (roads, railroads, subway). The weight of each road can be:
  - The length.
  - The expected time to traverse.
  - The expected cost to build.
- Example: a computer network, the weight of each edge (direct link) can be:
  - Latency.
  - Expected cost to build.



Minimum-Cost Spanning Tree (MST)

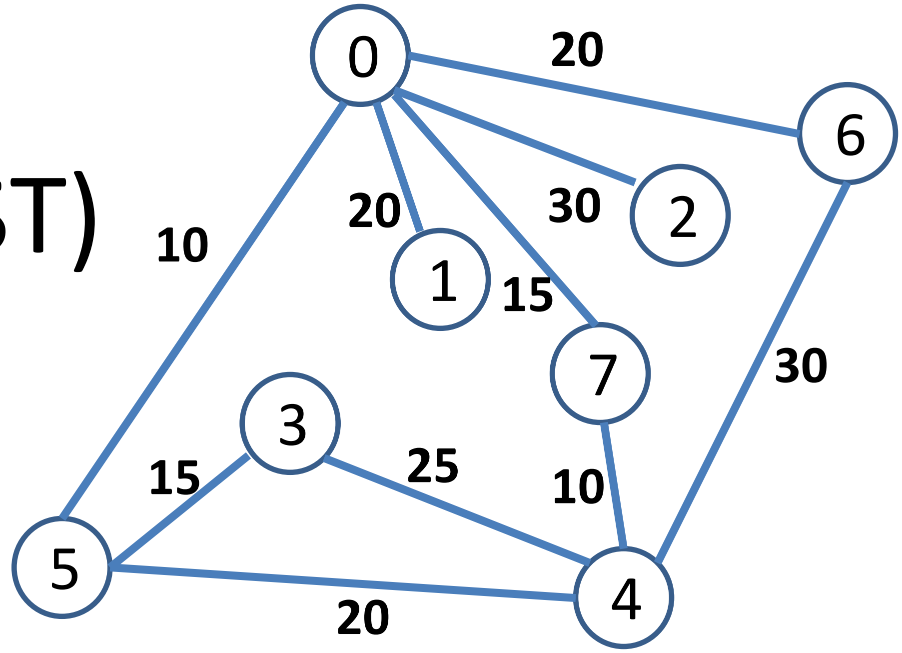
- Important problem in weighted graphs: finding a minimum-cost spanning tree:
- A tree that:
  - Connects **<u>all</u>** vertices of the graph.
  - Has the smallest possible total weight of edges.

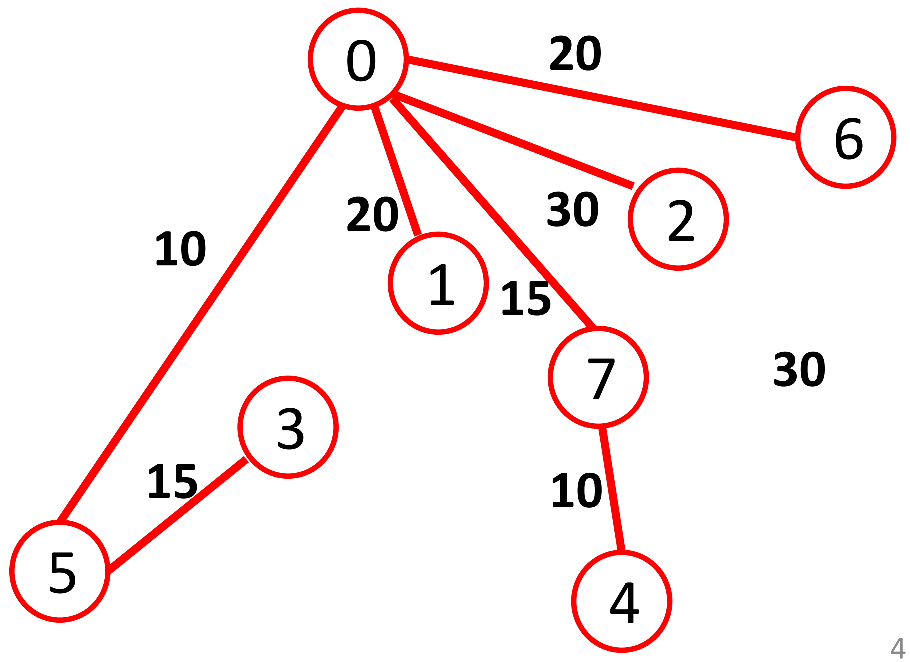




Minimum-Cost Spanning Tree (MST)

- We will only consider algorithms that compute the MST for **undirected graphs**.
- We will allow edges to have negative weights.
- Warning: later in the course (when we discuss Dijkstra's algorithm) we will need to make opposite assumptions:
  - Allow directed graphs.
  - Not allow negative weights.





Prim's Algorithm - Overview

- Prim's algorithm:
  - Start from an tree that contains a single vertex.
  - Keep growing that tree, by adding at each step the shortest edge connecting a vertex in the tree to a vertex outside the tree.
- As you see, it is a very simple algorithm, when stated abstractly.
- However, we have several choices regarding how to implement this algorithm.
- We will see three implementations, with **significantly different properties** from each other.

Prim's Algorithm - Simple Version

- Assume an adjacency matrix representation.
  - Each vertex is a number from 0 to V-1.
  - We have a V*V adjacency matrix ADJ, where: ADJ[v][w] is the weight of the edge connecting v and w.
  - If v and w are not connected, ADJ[v][w] = infinity.

Prim's Algorithm - Simple Version

1. Start by adding vertex 0 to the MST (minimum-cost spanning tree).
2. Repeat until all vertices have been added to the tree:
    3. From all edges connecting vertices from the current tree to vertices outside the current tree, select the smallest edge.
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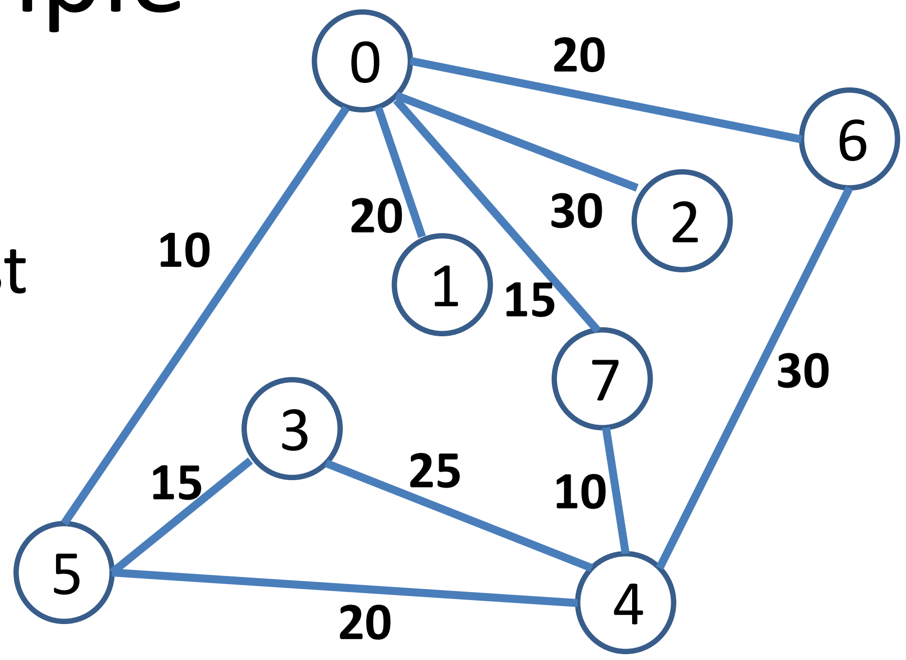

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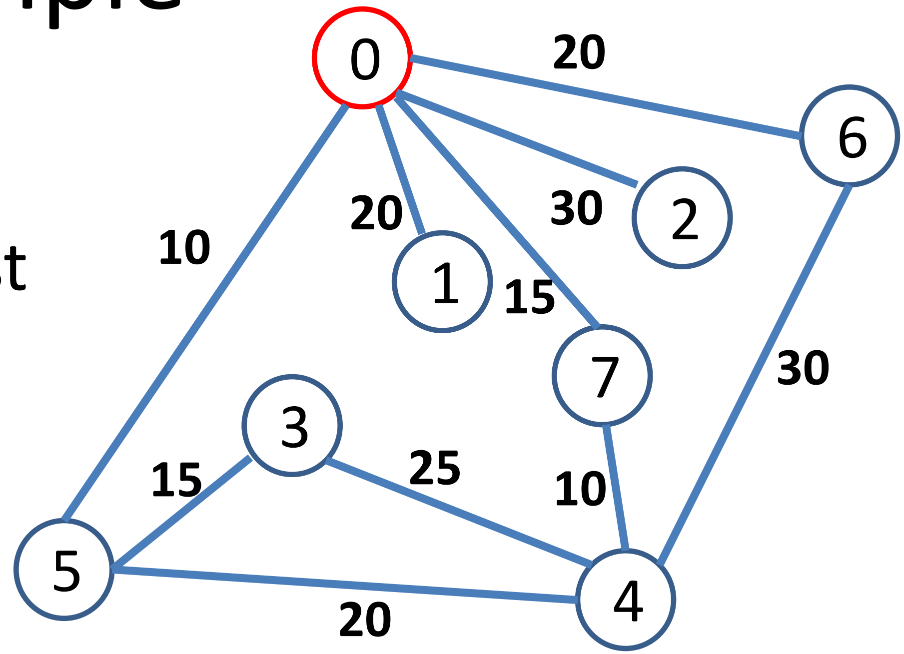

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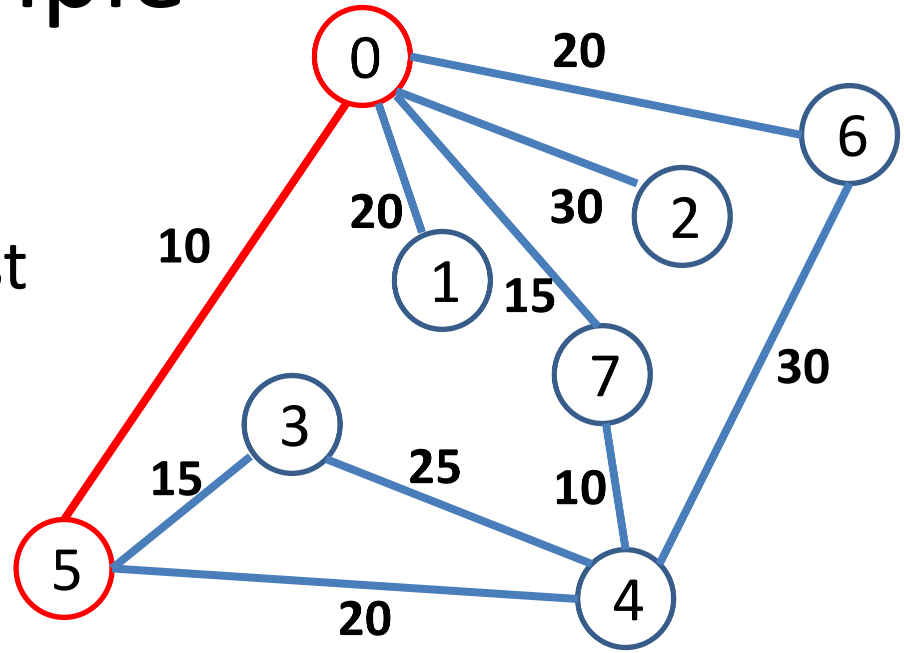

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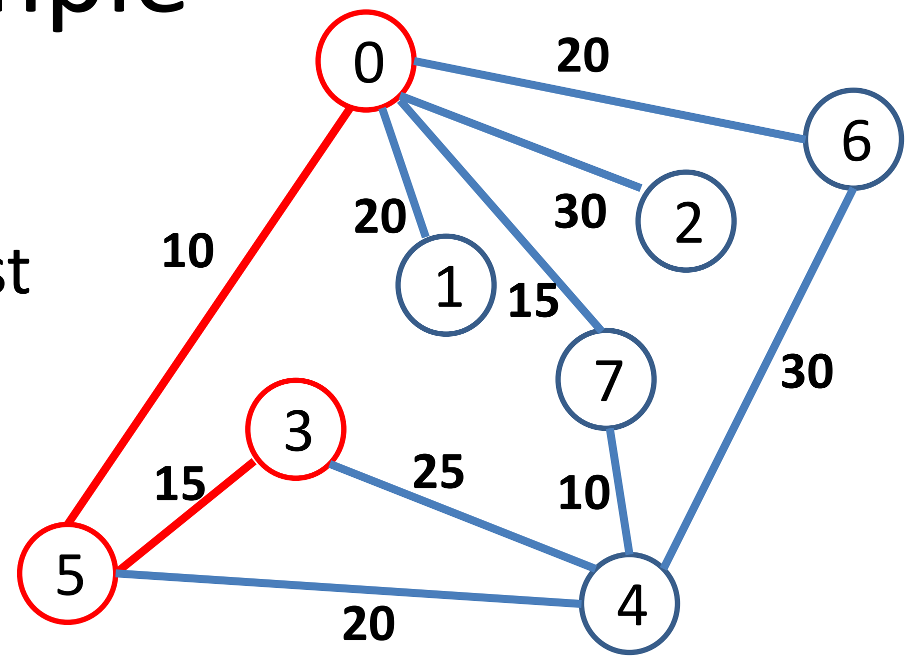

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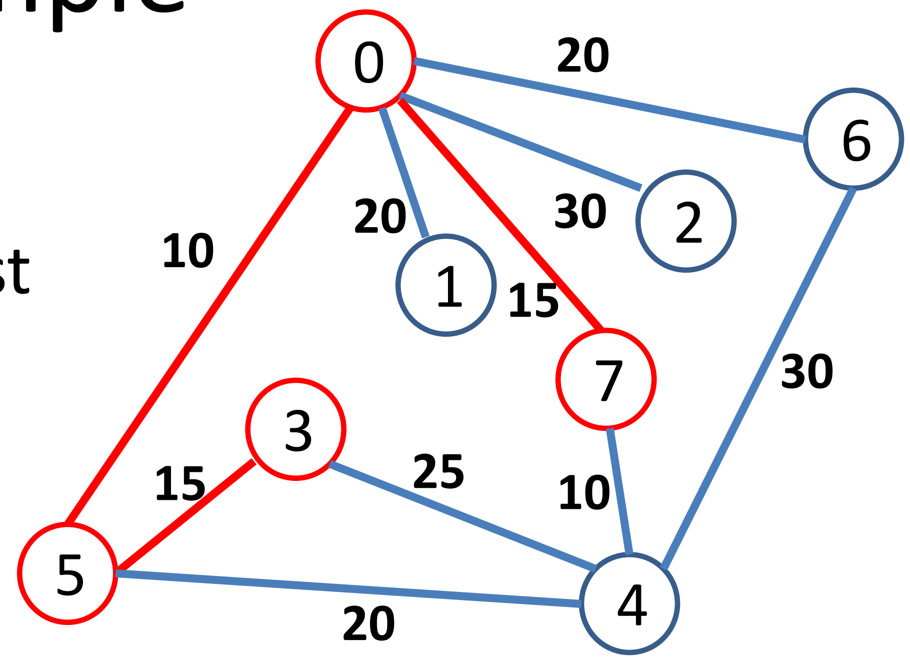

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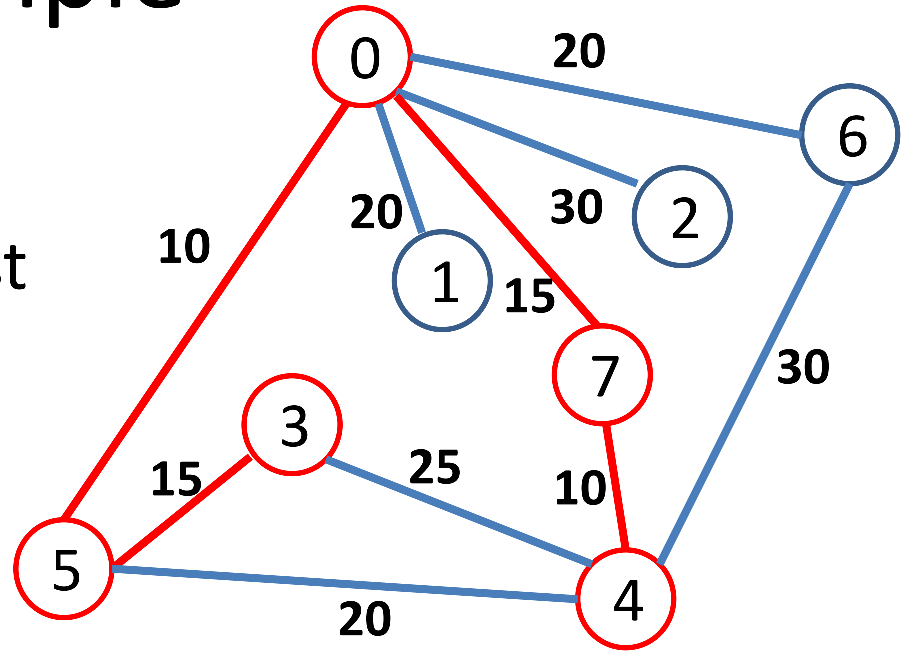

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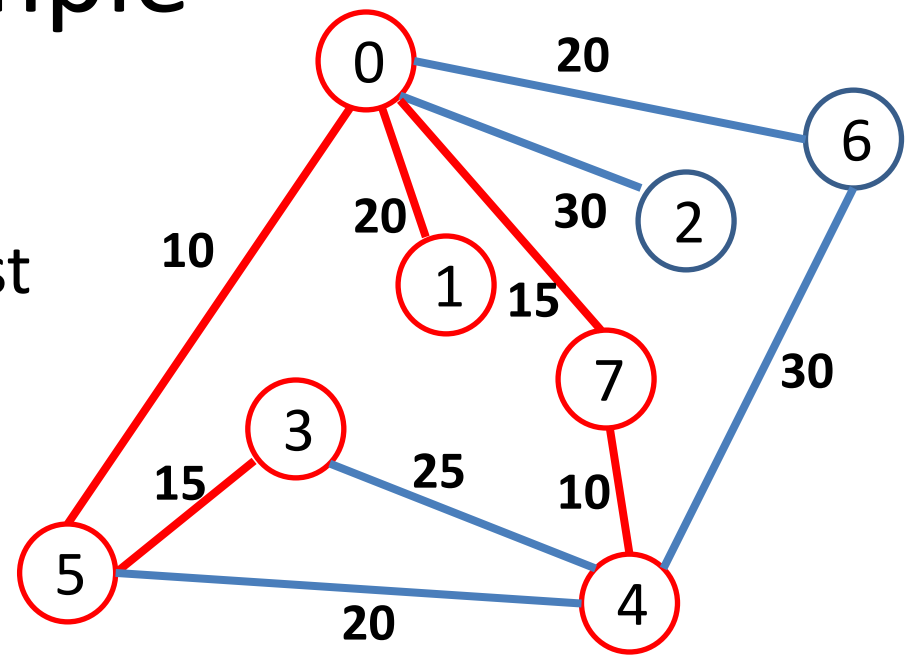

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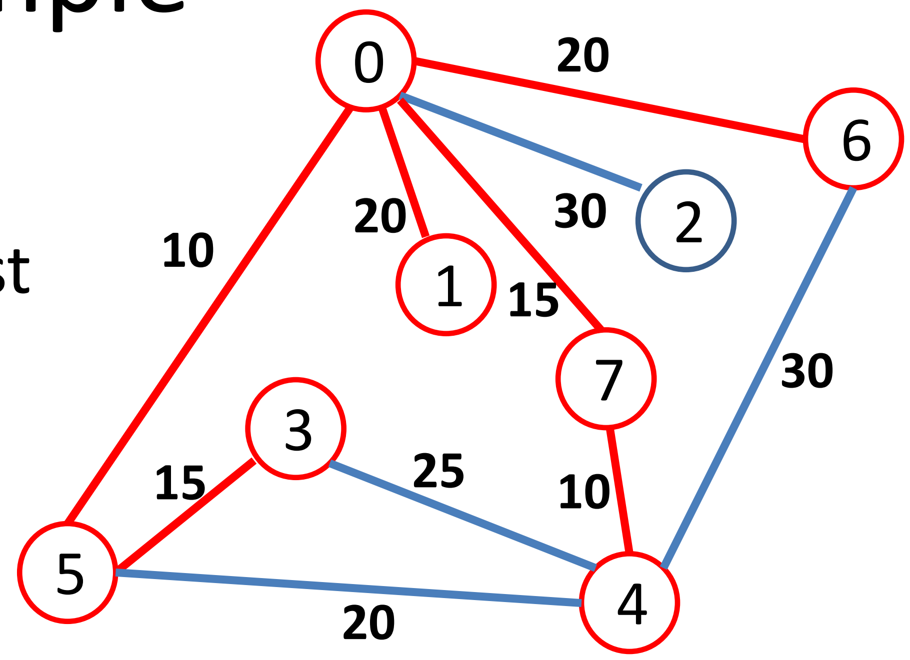

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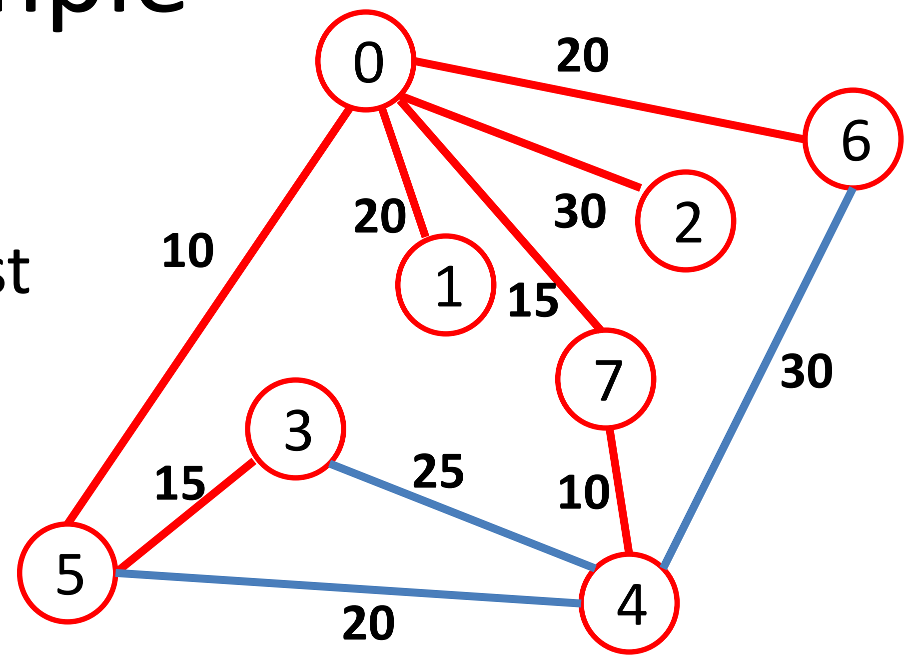

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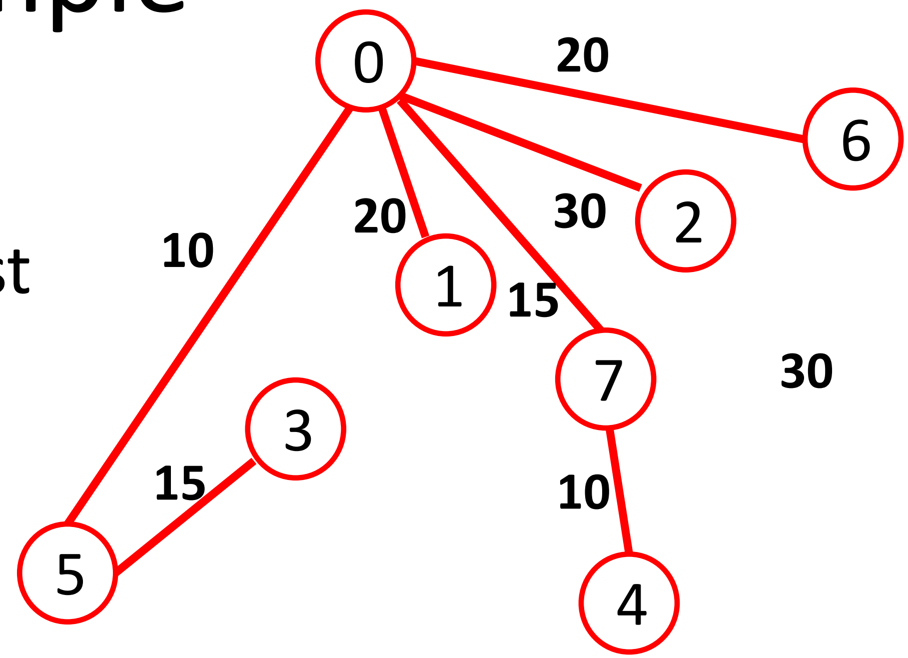

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- Running time?

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- Most naive implementation: time ???
  - Every time we add a new vertex and edge, go through all edges again, to identify the next edge (and vertex) to add.

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- Most naive implementation: time O(VE).
  – Every time we add a new vertex and edge, go through all edges again, to identify the next edge (and vertex) to add.

Prim's Algorithm - Dense Graphs

- A dense graph is nearly full, and thus has $O(V^2)$ edges.
  - For example, think of a graph where each vertex has at least V/2 neighbors.
- Just reading the input (i.e., looking at each edge of the graph once) takes $O(V^2)$ time.
- Thus, we cannot possibly compute a minimum-cost spanning tree for a dense graph in less than $O(V^2)$ time.
- Prim's algorithm can be implemented so as to take O(V2) time, which is optimal for dense graphs.

Prim's Algorithm - Dense Graphs

- Again, assume an **adjacency matrix** representation.
- Every time we add a vertex to the MST, we need to update, for each vertex W not in the tree:
  - The smallest edge wt[W] connecting it to the tree.
  - If no edge connects W to the tree, wt[W] = infinity.
  - The tree vertex fr[W] associated with the edge whose weight is wt[W].
- These quantities can be updated in O(V) time when adding a new vertex to the tree.
- Then, the next vertex to add is the one with the smallest wt[W].

Example

- Every time we add a vertex to the MST, we need to update, for each vertex W not in the tree:
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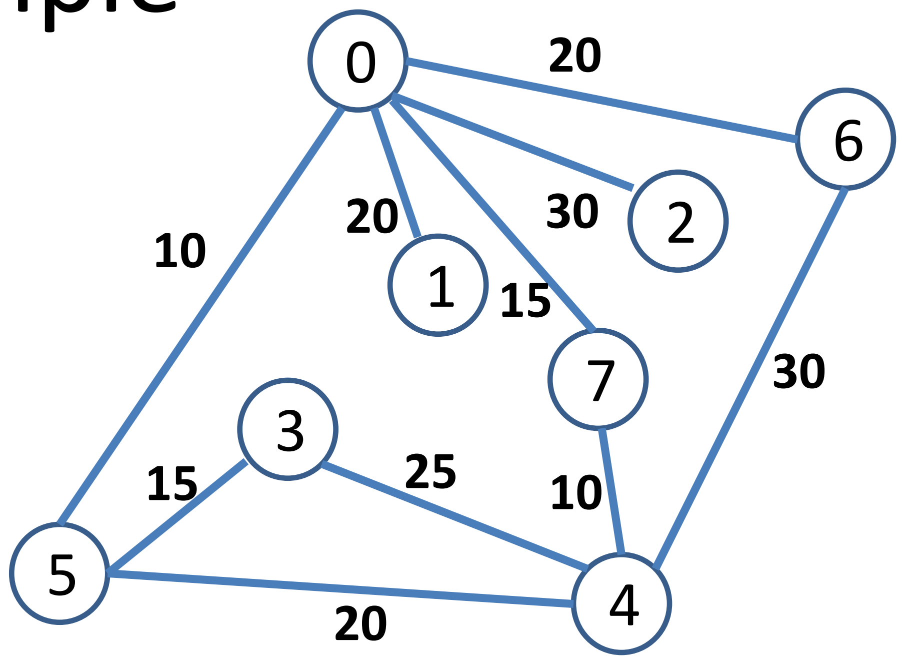

Prim's Algorithm: Dense Graphs

```
#define P G->adj[v][w]
void GRAPHmstV(Graph G, int st[], double wt[])
 { int v, w, min;
   for (v = 0; v < G->V; v++)
     { st[v] = -1; fr[v] = v; wt[v] = maxWT; }
   st[0] = 0; wt[G->V] = maxWT;
   for (min = 0; min != G->V; )
     {
       v = min; st[min] = fr[min];
       for (w = 0, min = G->V; w < G->V; w++)
         if (st[w] == -1)
           {
             if (P < wt[w])
               { wt[w] = P; fr[w] = v; }
             if (wt[w] < wt[min]) min = w;
           }}}
```

Prim's Algorithm - Dense Graphs

- Running time: ???

Prim's Algorithm - Dense Graphs

- Running time: $O(V^2)$
- Optimal for **dense graphs**.

Prim's Algorithm for Sparse Graphs

- A sparse graph is one that is not dense.
- This is somewhat vague.
- If you want a specific example, think of a case where the number of edges is linear to the number of vertices.
  - For example, if each vertex can only have between 1 and 10 neighbors, than the number of edges can be at most ???

Prim's Algorithm for Sparse Graphs

- A sparse graph is "one that is not dense".
- This is somewhat vague.
- If you want a specific example, think of a case where the number of edges is linear to the number of vertices.
  – For example, if each vertex can only have between 1 and 10 neighbors, than the number of edges can be at most 10*V.

Prim's Algorithm for Sparse Graphs

- If we use an adjacency matrix representation, then we can never do better than $O(V^2)$ time.
- Why?

Prim's Algorithm for Sparse Graphs

- If we use an adjacency matrix representation, then we can never do better than $O(V^2)$ time.
- Why?
  - Because just scanning the adjacency matrix to figure out where the edges are takes $O(V^2)$ time.
  - The adjacency matrix itself has size V*V.

Prim's Algorithm for Sparse Graphs

- If we use an adjacency matrix representation, then we can never do better than $O(V^2)$ time.
- Why?
  - Because just scanning the adjacency matrix to figure out where the edges are takes $O(V^2)$ time.
  - The adjacency matrix itself has size V*V.
- We have already seen an implementation of Prim's algorithm, using adjacency matrices, which achieves $O(V^2)$ running time.
- For sparse graphs, if we want to achieve better running time than $O(V^2)$, we have to switch to an adjacency lists representation.

Prim's Algorithm for Sparse Graphs

- Quick review: what exactly is an adjacency lists representation?

Prim's Algorithm for Sparse Graphs

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  - Each vertex is a number between 0 and V (same as for adjacency matrices).
  - The adjacency information is stored in an array ADJ of lists.
  - ADJ[w] is a list containing all neighbors of vertex w.
- What is the sum of length of all lists in the ADJ array?

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- What is the sum of length of all lists in the ADJ array?
  - 2*E (each edge is included in two lists).

Prim's Algorithm for Sparse Graphs

- For sparse graphs, we will use an implementation of Prim's algorithm based on:
  - A graph representation using **adjacency lists**.
  - A **priority queue** containing the set of edges on the fringe.
- An edge F will be included in this priority queue if: for some vertex w NOT in the tree yet, F is the shortest edge connecting w to vertex in the tree.

Prim's Algorithm - PQ Version

- Initialize a priority queue P.
- v = vertex 0
- While (true)
  - Add v to the spanning tree.
  - Let S = set of edges from v to vertices not yet in the tree to P:
  - If S is empty, exit.
  - For each F = (v, w) in S
    - If another edge F' in P also connects to w, keep the smallest of F and F'.
    - Else insert F to P.
  - F = remove_minimum(P)
  - v = vertex of F not yet in the tree.

Prim's Algorithm - PQ Version

- Running time???

Prim's Algorithm - PQ Version

- Running time? O(E lg V).

Kruskal's Algorithm: Overview

- Prim's algorithm works with a single tree, such that:
  – First, the tree contains a single vertex.
  – The tree keeps growing, until it spans the whole tree.
- Kruskal's algorithm works with a forest (a set of trees).
  – Initially, each tree in this forest is a single vertex.
  – Each vertex in the graph is its own tree.
  – We keep merging trees together, until we end up with a single tree.

Kruskal's Algorithm: Overview

1. Initialize a forest (a collection of trees), by defining each vertex to be its own separate tree.
2. Repeat until the forest contains a single tree:
    3. Find the shortest edge F connecting two trees in the forest.
    4. Connect those two trees into a single tree using edge F.

- As in Prim's algorithm, the abstract description is simple, but we need to think carefully about how exactly to implement these steps.

Kruskal's Algorithm: An Example

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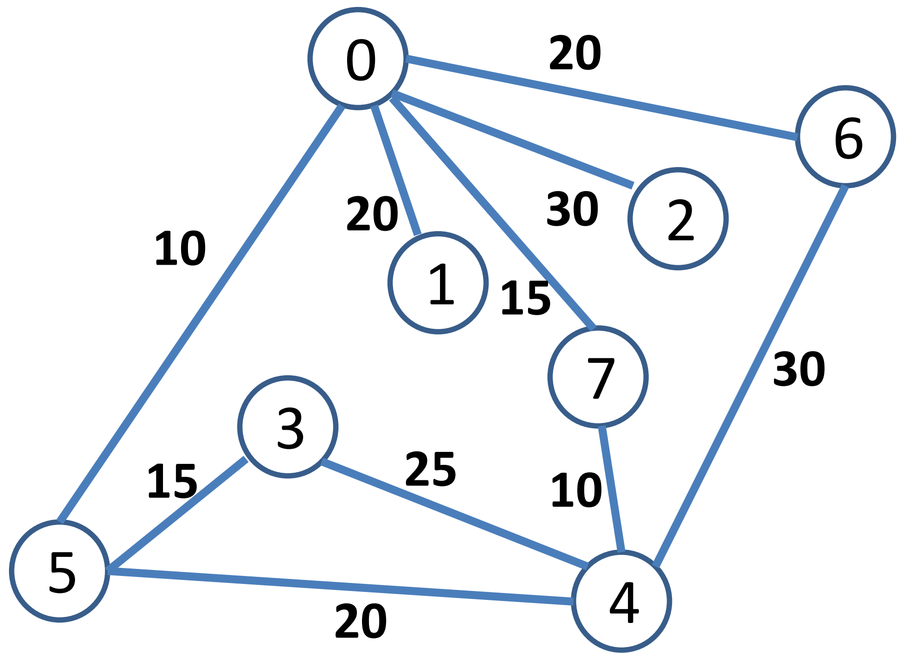

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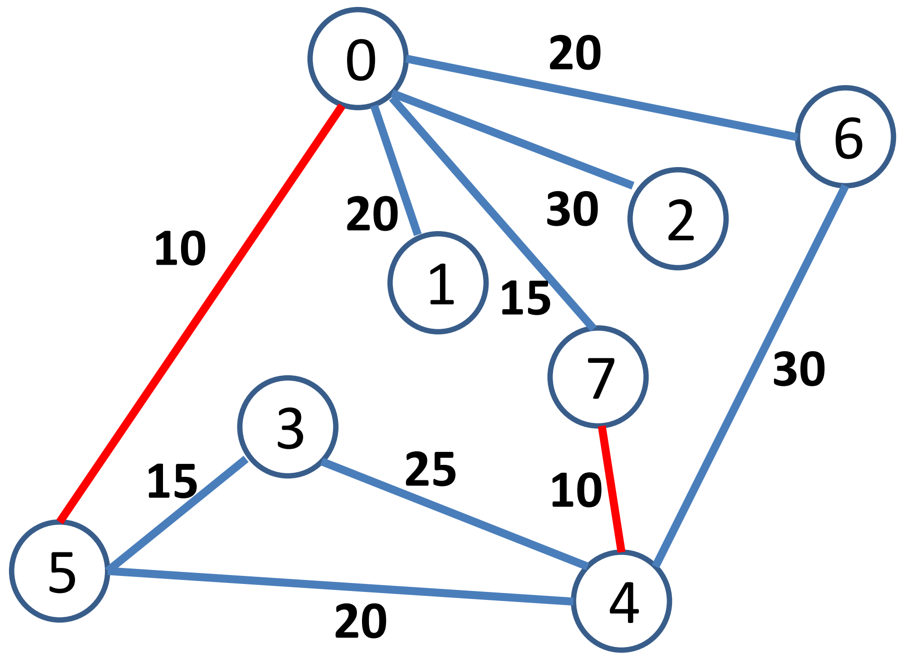

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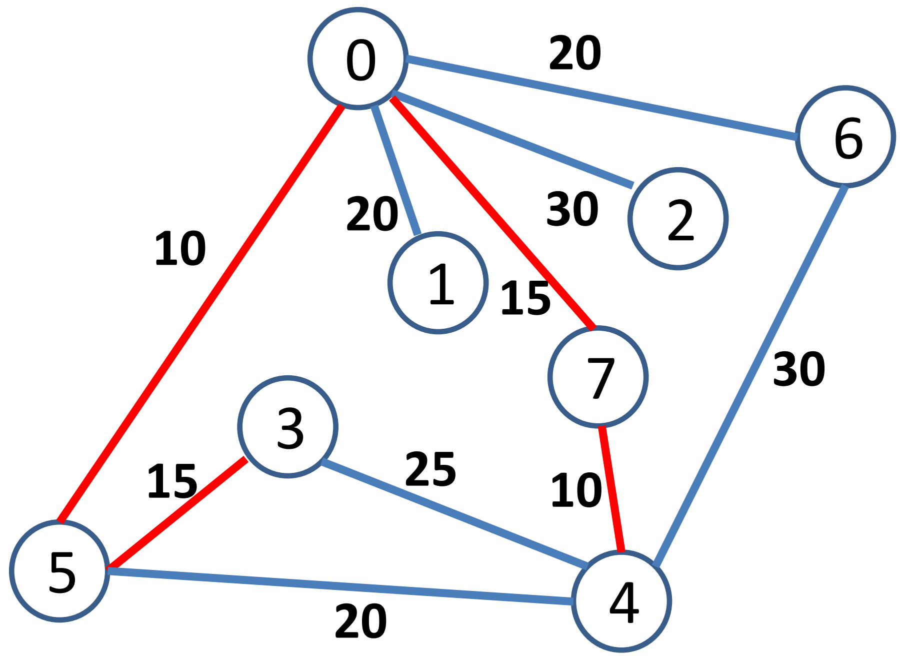

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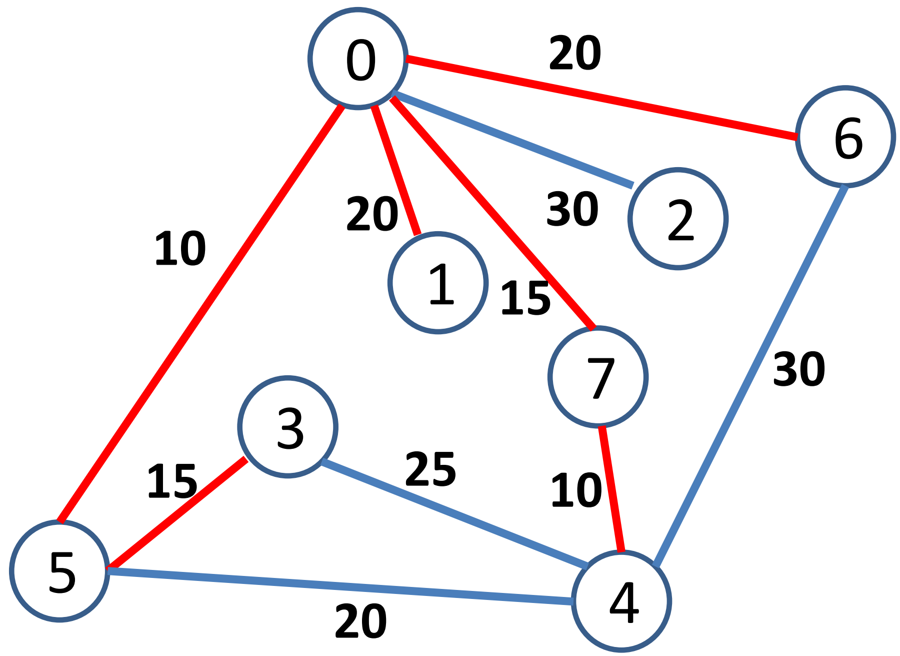

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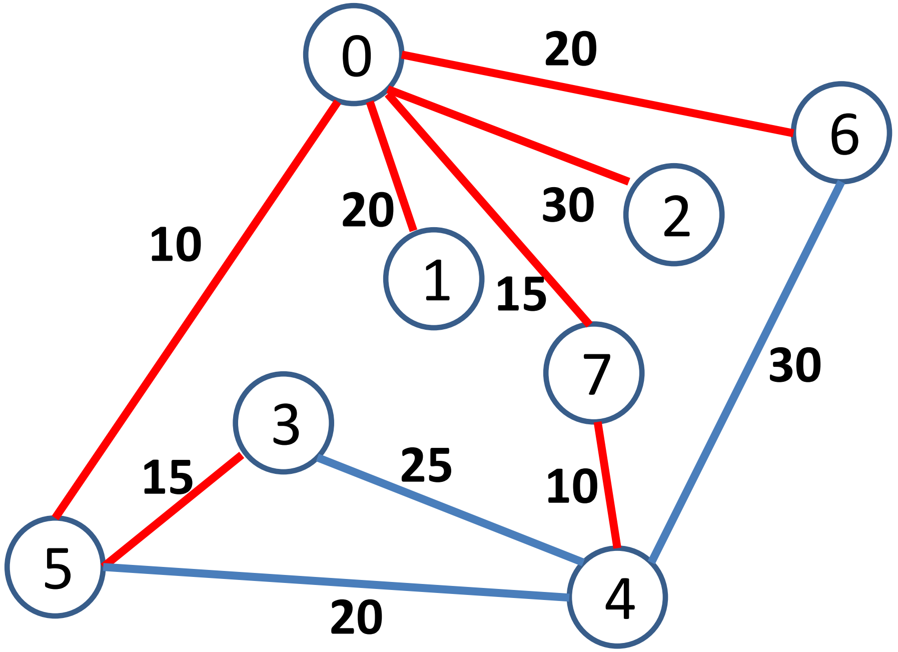

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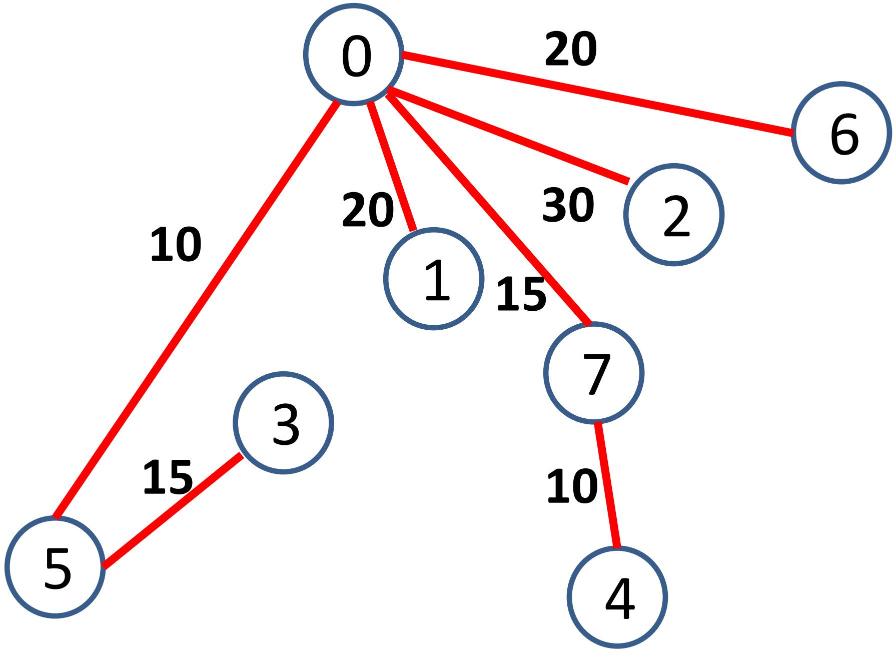

Kruskal's Algorithm: Simple Implementation

Assume graphs are represented usind **adjacency lists**.

1. Initialize a forest (a collection of trees), by defining each vertex to be its own separate tree.
   - How? We will use the same representation for forests that we used for union-find.
   - We will have an id array, where each vertex will point to its parent.
   - The root of each tree will be the ID for that tree.

- Time it takes for this step ???

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- Time it takes for this step? O(V)

Kruskal's Algorithm: Simple Implementation

2. Repeat until the forest contains a single tree:
   3. Find the shortest edge F connecting two trees in the forest.
   - Initialize F to some edge with infinite weight.
   - For each edge F' connecting (v, w):
     - Determine if v and w belong to the same tree in the forest.
     - If so, update F to be the shortest of F and F'.

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     - Determine if v and w belong to the same tree in the forest. **HOW?**
     - If so, update F to be the shortest of F and F'.

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Kruskal's Algorithm: Simple Implementation

2. Repeat until the forest contains a single tree:
   Total time for all iterations:
   3. Find the shortest edge F connecting two trees in the forest. **Time:**
   – Initialize F to some edge with infinite weight.
   – For each edge F' connecting (v, w):
     - Determine if v and w belong to the same tree in the forest. **HOW? By comparing find(v) with find(w). Time: O(lg V)**
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Kruskal's Algorithm: Simple Implementation

2. Repeat until the forest contains a single tree:
   Total time for all iterations: O(V*E*lg(V))
   3. Find the shortest edge F connecting two trees in the forest. **Time: O(E*lg(V))**
   – Initialize F to some edge with infinite weight.
   – For each edge F' connecting (v, w):
     - Determine if v and w belong to the same tree in the forest. **HOW? By comparing find(v) with find(w). Time: O(lg V)**
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Running Time for Simple Implementation

1. Initialize a forest (a collection of trees), by defining each vertex to be its own separate tree.
2. Repeat until the forest contains a single tree:
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- Running time for simple implementation: O(V*E*lg(V)).

Kruskal's Algorithm: Faster Version

1. Sort all edges, save result in array K.
2. Initialize a forest (a collection of trees), by defining each vertex to be its own separate tree.
3. For each edge F in K (in ascending order).
    4. If F is connecting two trees in the forest:
        5. Connect the two trees with F.
        6. If the forest is left with a single tree, break (we are done).

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Kruskal's Algorithm: Faster Version

1. Sort all edges, save result in array K. **Time: O(E lg E)**
2. Initialize a forest (a collection of trees), by defining each vertex to be its own separate tree. **Time: O(V)**
3. For each edge in K (in ascending order). **Time: O(E lg V)**
    4. If F is connecting two trees in the forest:
       Time? O(lg V), two find operations
        5. Connect the two trees with F. **Time: O(1), union operation**
        6. If the forest is left with a single tree, break (we are done).

- Overall running time???

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- Overall running time: O(E lg E).

Kruskal's Algorithm - PQ Version

- In the previous implementation, we sort edges at the beginning.
  - This takes O(E lg E) time, which dominates the running time of the algorithm.
  - Thus, the entire algorithm takes O(E lg E) time.
- We can do better if, instead of sorting all edges at the beginning, we instead insert all edges into a priority queue.
  - How long does that take, if we use a heap?

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- We can do better if, instead of sorting all edges at the beginning, we instead insert all edges into a priority queue.
  - How long does that take, if we use a heap?
  - O(E) time.
- We can also do better if, for the find operation, we use the most efficient version discussed in the textbook.
  - That version flattens paths that it traverses.
  - Running time: O(lg* V).
  - lg*(V) is the number of times we need to apply lg to V to obtain 1.

Detour: lg*

- lg*(2) = ?
- lg*(4) = ?
- lg*(16) = ?

Detour: lg*

- lg*(2) = 1, because lg(2) = 1.
- lg*(4) = 2, because lg(lg(4)) = 1.
- lg*(16) = 3, because lg(lg(lg(16))) = 1.
- lg*(???) = 4
- lg*(???) = 5

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- lg*(4) = 2, because lg(lg(4)) = 1.
- lg*(16) = 3, because lg(lg(lg(16))) = 1.
- lg*(65536) = 4, because lg(65536) = 16.
- lg*(2^{65536}) = 5, because lg(2^{65536}) = 65536.

- I don't expect we will get to deal with data sizes larger than 2^{65536} in our lifetime.
- Thus, lg* effectively has 5 as an upper bound, so for practical purposes we can treat it as a constant.

Kruskal's Algorithm - PQ Version

1. Initialize a heap with the edges (using weight as key).

2. Initialize a forest (a collection of trees), by defining each vertex to be its own separate tree.
3. While (true).
    4. F = remove_mininum(heap).
    5. If F is connecting two trees in the forest:

        6. Connect the two trees with F.
        7. If the forest is left with a single tree, break (we are done).

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2. Initialize a forest (a collection of trees), by defining each vertex to be its own separate tree. **Time?**
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    5. If F is connecting two trees in the forest: **Time?**
        6. Connect the two trees with F. **Time?**
        7. If the forest is left with a single tree, break (we are done).

- **Overall running time?**

Kruskal's Algorithm - PQ Version

1. Initialize a heap with the edges (using weight as key). **Time? O(E)**
2. Initialize a forest (a collection of trees), by defining each vertex to be its own separate tree. **Time? O(V)**
3. While (true). **Time? X lg V.** **X: number of iterations.**
    4. F = remove_mininum(heap). **Time? O(lg E)**
    5. If F is connecting two trees in the forest: **Time? O(lg* V), find operation**
        6. Connect the two trees with F. **Time? O(1)**
        7. If the forest is left with a single tree, break (we are done).

- **Overall running time? E + X lg V.**

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        6. Connect the two trees with F. **Time? O(1)**
        7. If the forest is left with a single tree, break (we are done).

- Overall running time? E + X lg V.
  - X is the number of edges in the graph with weight <= the maximum weight of an edge in the final MST.
  - $E < V^2$, so $\lg E < 2 \lg V$, so $O(\lg E) = O(\lg V)$.