# Chapter 1.3 Quantifiers, Predicates, and Validity

Reading: 1.3

Next Class: 1.4



## Motivation

- Propositional logic allows to translate and prove certain arguments from natural language

  – If John's wallet was stolen, then the thief must have been in his office. If the thief was in his office, then Jack can not be the thief. If Jack is not the thief, then the wallet was not stolen. Therefore the wallet was not stolen.

  $$(A \to B) \land (B \to C') \land (C' \to A') \to A'$$

- But: Not all arguments can be translated into propositional logic. Very often arguments for propositional logic sound awkward and we would rather formulate them differently.

  – If John's wallet was stolen, then the thief was in the office. Only Jack could have stolen the wallet but Jack was not in the office. Therefore the wallet was not stolen.

# Motivation

- Propositional logic has problems expressing many aspects of natural language **because it is limited to fixed statements**.

  – Propositional logic has no notion of sequence or interrelation between statements except in the form of explicit compound statements. Either propositions are identical or they are independent.

  – Propositional logic does not provide a means to express quantities or numbers.

  Eg. "For every $x$, $x > 0$."





# Variables and Statements

- Variables in Logic

  A variable is a symbol that stands for an individual in a collection or set. For example, the variable $x$ may stand for one of the days. We may let $x$ = Monday or $x$ = Tuesday, etc.
- Like in mathematics, variables here allow to indicate that different parts refer to the same object. Variables are often represented by lowercase characters from the end of the alphabet $(x, y, \ldots)$.
- Constants are used much in the same way but they refer to one specific, pre-defined object. Constants are often represented by lowercase characters from the beginning of the alphabet $(a, b, \ldots)$.

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# Variables and Statements

- Incomplete Statements

  A sentence containing a variable is called an incomplete statement. An incomplete statement is about the individuals in a definite domain or set. When we replace the variable by the name of an individual in the set we obtain a statement about that individual.
  - Example of an incomplete statement : "*x* has 30 days."
    - Here, *x* can be any month and substituting that, we will get a complete statement.



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# Predicates

- Predicate

  It is the verbal statement that describes the property of a variable. Usually represented by the letter P, the notation P($x$) is used to represent some unspecified property or predicate that $x$ may have
  - statement A — E.g. The earth is round..
  - unary predicate $P(x)$ — E.g. $x$ is a person.
  - binary predicate $F(x; y)$ — E.g. $x$ is the father of $y$.
  - n-ary predicate $S(x_1, \ldots, x_n)$

- The collection of objects that satisfy the property P($x$) is called the *domain of interpretation.*
- The value of a predicate can be either *TRUE, FALSE,* or *undefined* if the identity of the object(s) is unknown.

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# Quantifiers

- Quantifiers:

  Quantifiers are phrases that refer to given quantities, such as "for some" or "for all" or "for every," *indicating how many objects have a certain property.*

- Two kinds of quantifiers: Universal and Existential
- Universal Quantifier: represented by ∀

  The symbol is translated as and means "For all", "Given any", "for each," "Any object,"or "for every," and is known as the universal quantifier.
  - Specifies all objects in the domain of interpretation
  - The universal quantifier allows to translate more general expressions.
    - Bill is tall. All jockeys are not tall. Therefore Bill is not a jockey.

      $T(b) \wedge (\forall x)(J(x) \rightarrow [T(x)]') \rightarrow J(b)'$



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- Existential Quantifier: represented by ∃

  The symbol is translated as and means variously "for some," "there exist(s)," "there is a," or "for at least one".
  - Specifies the existence of at least one object within the domain of interpretation.
  - Eg. $(\exists x)(x=2)$
  - Important: $(\exists x)$ does not mean that there is exactly one object but rather that there is at least one object.
  - The existential quantifier allows to translate arguments like this:

    Bill is tall. There are no tall jockeys. Therefore Bill is not a jockey.

    $T(b) \wedge [(\exists x)(J(x) \wedge T(x))]' \rightarrow [J(b)]'$

# Quantifiers and Predicates

- Combining the quantifier and the predicate, we get a complete statement of the form $(\forall x)P(x)$ or $(\exists x)P(x)$.
- Truth value of expressions formed using quantifiers and predicates
  - What is the truth value of $(\forall x)P(x)$ where $x$ is all the months and $P(x) = x$ has less than 32 days.

    Undoubtedly, the above is true since no month has 32 days.
- Using variables and predicates many additional natural language expressions can be represented in formal logic.
  - Bill is a tall person. If a person is tall that person can not be a jockey. Therefore Bill can not be a jockey.
  - Let $b$ is Bill, $P(x)$ means $x$ is a person, $T(x)$ means $x$ *is tall*, $J(x)$ means $x$ *is* a jockey, we have

    $$(P(b) \wedge T(b)) \wedge [(\forall x)((P(x) \wedge T(x))] \rightarrow [J(x)]') \rightarrow [J(b)]'$$

    the domain of interpretation is the whole universe of objects.





# Truth value of the following expressions

- Truth of expression $(\forall x)P(x)$
  1. $P(x)$ is the property that $x$ is yellow, and the domain of interpretation is the collection of all flowers: not true
  2. $P(x)$ is the property that $x$ is a plant, and the domain of interpretation is the collection of all flowers: true
  3. $P(x)$ is the property that $x$ is positive, and the domain of interpretation consists of integers: not true
  - Can you find one interpretation in which both $(\forall x)P(x)$ is true and $(\exists x)P(x)$ is false? Not possible
  - Can you find one interpretation in which both $(\exists x)P(x)$ is true and $(\forall x)P(x)$ is false? Case 1 as mentioned above
- Predicates involving properties of single variables : unary predicates
- Binary, ternary and $n$-ary predicates are also possible.
  - $(\forall x)(\exists y)Q(x,y)$ is a binary predicate. This expression reads as "for every $x$ there exists a $y$ such that $Q(x,y)$."

## 10 Interpretation

- Formal definition: An interpretation for an expression involving predicates consists of the following:
  1. A collection of objects, called the domain of interpretation, which must include at least one object.
  2. An assignment of a property of the objects in the domain to each predicate in the expression.
  3. An assignment of a particular object in the domain to each constant symbol in the expression.
- **Predicate wffs** can be built similar to **propositional wffs** using logical connectives with predicates and quantifiers.
- Examples of predicate wffs
  - $(\forall x)[P(x) \to Q(x)]$
  - $(\forall x)\ ((\exists y)[P(x,y) \vee Q(x,y)] \to R(x))$
  - $S(x,y) \wedge R(x,y)$



## 11 Scope of a variable in an expression

- Brackets are used wisely to identify the **scope of the variable**, the section of the wff to which the quantifier applies.
  - $(\forall x)\ [(\exists y)[P(x,y) \vee Q(x,y)] \to R(x)]$
    Scope of $(\exists y)$ is $P(x,y) \vee Q(x,y)$ while the scope of $(\forall x)$ is the entire expression.
  - $(\forall x)S(x) \vee (\exists y)R(y)$
    Scope of $x$ is $S(x)$ while the scope of $y$ is $R(y)$.
  - $(\forall x)[P(x,y) \to (\exists y)\ Q(x,y)]$
    Scope of variable $y$ is not defined for $P(x,y)$ hence $y$ is called a free variable. Such expressions might not have a truth value at all.
- What is the truth of the expression
  - $(\exists x)[A(x) \wedge (\forall y)[B(x,y) \to C(y)]]$ is the interpretation
  - $A(x)$ is "$x > 0$" , $B(x, y)$ is "$x > y$" and $C(y)$ is "$y \leq 0$" where the domain of $x$ is positive integers and the domain of y is all integers

True, x=1 is a positive integer and any integer less than $x$ is $\leq 0$

## Translation: Verbal statements to symbolic form

- In natural language there are many ways to express the same thing. This makes translation into formal logic more difficult.
- “Every person is nice” can be rephrased as “For any thing, if it is a person, then it is nice.” So, if P($x$) is “$x$ is a person” and Q($x$) be “$x$ is nice,” the statement can be symbolized as
  - $(\forall x)[P(x) \rightarrow Q(x)]$
  - “All persons are nice” or “Each person is nice” will also have the same symbolic form.





## Translation

- “There is a nice person” can be rewritten as “There exists something that is both a person and nice.”
  - In symbolic form, $(\exists x)[P(x) \land Q(x)]$.
  - Variations: “Some persons are nice” or “There are nice persons.”
- What would the following form mean for the example above?

$$(\exists x)[P(x) \rightarrow Q(x)]$$

  - This will only be true if there are no persons in the world but that is not the case.
  - Hence such a statement is false, so almost always,
    $\exists$ goes with $\land$ (conjunction) and
    $\forall$ goes with $\rightarrow$ (implication).

# Translation

- Hint: Avoid confusion by framing the statement in different forms as possible.

- The word "only" can be tricky depending on its presence in the statement.
  - X loves only Y ⇔ If X loves anything, then that thing is Y.
  - Only X loves Y ⇔ If anything loves Y, then it is X.
  - X only loves Y ⇔ If X does anything to Y, then it is love.





# Translation

- Example for forming symbolic forms from predicate symbols
  - D($x$) is "$x$ is dog"; R($x$) is "$x$ is a rabbit"; C($x,y$) is "$x$ chases $y$"
  - All dogs chase all rabbits ⇔
    For anything, if it is a dog, then for any other thing, if it is a rabbit, then the dog chases it ⇔

  - Some dogs chase all rabbits ⇔
    There is something that is a dog and for any other thing, if that thing is a rabbit, then the dog chases it ⇔

  - Only dogs chase rabbits ⇔
    For any two things, if one is a rabbit and the other chases it, then the other is a dog ⇔
    All things that chase rabbits are dogs.

## Negation of statements

- $[(\forall x)A(x)]'$ Not for all objects $x$ ... / At least one object $x$ is not ...
  $(\forall x)[A(x)]'$ For all objects $x$ it is not true that ... / No object $x$ ...
  $[(\exists x)A(x)]'$ There exists no $x$ such that ... / No object $x$ ...
  $(\exists x)[A(x)]'$ There exists an object $x$ such that not ... / Not for all objects $x$ ...
- *Eg: A(x): Everything is fun*
  Negation will be "it is false that everything is fun," i.e. "something is no fun."
- In symbolic form, $[(\forall x)A(x)]' \Leftrightarrow (\exists x)[A(x)]'$
- Eg: Similarly negation of "*Something is fun*" is "Nothing is fun" or "Everything is boring."
- Hence, $[(\exists x)A(x)]' \Leftrightarrow (\forall x)[A(x)]'$





## Class Exercise

- What is the negation of "Everybody loves somebody sometime."
  - Everybody hates somebody sometime
  - Somebody loves everybody all the time
  - Everybody hates everybody all the time
  - Somebody hates everybody all the time √

- What is the negation of the following statements?
- Some pictures are old and faded.
  Every picture is neither old nor faded.
- All people are tall and thin.
  Someone is short or fat.
- Some students eat only pizza.
  Every student eats something that is not pizza.
- Only students eat pizza.
  There is a non-student who eats pizza.

# Class exercise

- S($x$): $x$ is a student; I($x$): $x$ is intelligent; M($x$): $x$ likes music
- Write wffs that express the following statements:
  - All students are intelligent.

    For anything, if it is a student, then it is intelligent ⇔

  - Some intelligent students like music.

    There is something that is intelligent and it is a student and it likes music ⇔

  - Everyone who likes music is a stupid student.

    For anything, if that thing likes music, then it is a student and it is not intelligent ⇔

  - Only intelligent students like music.

    For any thing, if it likes music, then it is a student and it is intelligent ⇔





# Validity

- Analogous to a tautology of propositional logic.
- Truth of a predicate wff depends on all possible interpretations.
- A predicate wff is valid if it is true in all possible interpretations just like a propositional wff is true if it is true for all rows of the truth table.
- A valid predicate wff is intrinsically true.

| | Propositional Wffs | Predicate Wffs |
|---|---|---|
| Truth values | True or false – depends on the truth value of statement letters | True, false or neither (if the wff has a free variable) |
| Intrinsic truth | Tautology – true for all truth values of its statements | Valid wff – true for all interpretations |
| Methodology | Truth table to determine if it is a tautology | No algorithm to determine validity |

# Validity examples

- $(\forall x)P(x) \rightarrow (\exists x)P(x)$
  - This is valid because if every object of the domain has a certain property, then there exists an object of the domain that has the same property.
- $(\forall x)P(x) \rightarrow P(a)$
  - Valid – quite obvious since is a member of the domain of $x$.
- $(\exists x)P(x) \rightarrow (\forall x)P(x)$
  - Not valid since the property cannot be valid for all objects in the domain if it is valid for some objects of than domain. Can use a mathematical context to check as well.
  - Say $P(x)$ = "$x$ is even," then there exists an integer that is even but not every integer is even.
- $(\forall x)[P(x) \vee Q(x)] \rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$
  - Invalid, can prove by mathematical context by taking $P(x) = x$ is even and $Q(x) = x$ is odd.
  - In that case, the hypothesis is true but the conclusion is false because it is not the case that every integer is even or that every integer is odd.





# Class Exercise

- What is the truth of the following wffs where the domain consists of integers:
  - $(\forall x)[L(x) \rightarrow O(x)]$ where $O(x)$ is "x is odd" and L(x) is "x < 10"?
  - $(\exists y)(\forall x)(x + y = 0)$?
  - $(\exists y)(\exists x)(x^2 = y)$?
  - $(\forall x)[x < 0 \rightarrow (\exists y)(y > 0 \wedge x + y = 0)]$?

- Using predicate symbols and appropriate quantifiers, write the symbolic form of the following English statement:
  - $D(x)$ is "$x$ is a day", M is "Monday", T is "Tuesday"
    $S(x)$ is "$x$ is sunny", $R(x)$ is "$x$ is rainy"
  - Some days are sunny and rainy.
  - It is always a sunny day only if it is a rainy day.
  - It rained both Monday and Tuesday.
  - Every day that is rainy is not sunny.