# Chapter 4.3 Principles of Inclusion and Exclusion

Read: 4.3

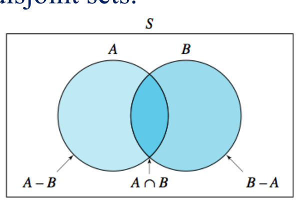
Next Class: 4.4





## Principle of Inclusion & Exclusion

- If A and B are subsets of universal set S, then (A-B), (B-A) and ($A \cap B$) are disjoint sets.




- As seen from the figure, (A-B) $\cup$ (B-A) $\cup$ ($A \cap B$) is the same as $A \cup B$.

- For three disjoint sets (based on the addition principel)

  $|(A-B) \cup (B-A) \cup (A \cap B)| = |A-B| + |B-A| + |A \cap B|$

## Principle of Inclusion & Exclusion



- We have proved in last class that for two finite sets A and B (don't need to be disjoint)

  $|A-B| = |A| - |A \cap B|$

  Hence, using this, we get

  $|(A-B) \cup (B-A) \cup (A \cap B)| = |A| - |A \cap B| + |B-A| + |A \cap B|$

  $= |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B|$

  Hence, $|A \cup B| = |A| + |B| - |A \cap B|$



## Principle of Inclusion & Exclusion



$|A \cup B| = |A| + |B| - |A \cap B|$

- The above equation represents the principle of inclusion and exclusion for two sets A and B.
- The name comes from the fact that to calculate the elements in a union, we include the individual elements of A and B but subtract the elements common to A and B so that we don't count them twice.
- This principle can be generalized to *n* sets.

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## Example: Inclusion and Exclusion Principle

- Example 1: How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?
  - Let us assume that A = set of all integers from 1 to 1000 that are multiples of 3.
  - Let us assume that B = set of all integers from 1 to 1000 that are multiples of 5.
  - $A \cup B$ = The set of all integers from 1 to 1000 that are multiples of either 3 or 5.
  - $A \cap B$ = The set of all integers that are both multiples of 3 and 5, which also is the set of integers that are multiples of 15.



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## Example: Inclusion/exclusion principle

To use the inclusion/exclusion principle to obtain $|A \cup B|$, we need $|A|$, $|B|$ and $|A \cap B|$.

- From 1 to 1000, every third integer is a multiple of 3, each of this multiple can be represented as $3p$, for any integer $p$ from 1 through 333, Hence $|A| = 333$.
- Similarly for multiples of 5, each multiple of 5 is of the form $5q$ for some integer $q$ from 1 through 200. Hence, we have $|B| = 200$.
- To determine the number of multiples of 15 from 1 through 1000, each multiple of 15 is of the form $15r$ for some integer $r$ from 1 through 66.
- Hence, $|A \cap B| = 66$.

From the principle, we have the number of integers either multiples of 3 or multiples of 5 from 1 to 1000 given by

$$|A \cup B| = 333 + 200 - 66 = 467.$$

## Example: Inclusion/exclusion principle for 3 sets

- **Example 2**: In a class of students undergoing a computer course the following were observed.
  - Out of a total of 50 students: 30 know Python, 18 know Java, 26 know C++, 9 know both Python and Java, 16 know both Python and C++, 8 know both Java and C++, 47 know at least one of the three languages.

From this we have to determine

- a. How many students know none of these languages?
- b. How many students know all three languages?





## Example: Inclusion/exclusion principle for 3 sets

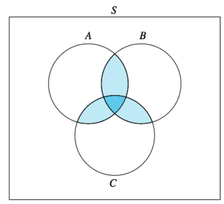
a. We know that 47 students know at least one of the three languages in the class of 50. The number of students who do not know any of three languages is given by the difference between the number of students in class and the number of students who know at least one language.
   - Hence, the students who know none of these languages = 50 – 47 = 3.

b. Students know all three languages, so we need to find $|A \cap B \cap C|$.
   - A = All the students who know Python in class.
   - B = All the students who know C++ in the class.
   - C = All the students who know Java in class.

- We have to derive the inclusion/exclusion formula for three sets
  $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$

## Example: Inclusion/exclusion principle for 3 sets

- $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$
  $= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$
  $= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |A \cap B \cap C|)$
  $\mathbf{= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|}$
- Given in the problem are the following:
  $|B \cap C| = 8$
  $|A \cap B| = 9$
  $|A \cap C| = 16$
  $|A \cup B \cup C| = 50$




- Hence, using the above formula, we have
  $47 = 30 + 26 + 18 - 9 - 16 - 8 + |A \cap B \cap C|$
  Hence, $|A \cap B \cap C| = 6$





## Principle of Inclusion and Exclusion

Given the finite sets $A_1, \ldots \ldots, A_n$, $n \geq 2$, then

$$|A_1 \cup \ldots\ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap \ldots \cap A_n|$$

- In this above equation, the term $\sum_{1 \le i \le j \le n} |A_i \cap A_j|$ says add together elements in all intersections of the form $A_i \cap A_j$ where $i < j$.
- What are these terms for $n = 3$? this gives $A_1 \cap A_2$ ($i$=1, $j$=2), $A_1 \cap A_3$ ($i$=1, $j$=3), and $A_2 \cap A_3$ ($i$=2, $j$=3). This agrees with the proof of inclusion/exclusion principle for three sets in the previous slide, where $A_1 = A$, $A_2 = B$, and $A_3 = C$.

## Principle of Inclusion and Exclusion (cont'd)

- The principle of inclusion and exclusion calculates for given the sets of events $A_1, \ldots, A_n$ , the total number of events "$A_1$ OR…,OR $A_n$".
- Note: The addition principle is a special case of this principle where all the sets of events are disjoint.
- $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C|$
  $-|A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| +$
  $|A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$
- Example: If you can select between the 3 kitchen tools {pan, spoon, scissors} and 4 home-working tools {glue, nails, scissors, hammer}, you have a total number of 3+4-1 = 6 choices.





## Pigeonhole Principle

- If more than $k$ items are placed into $k$ bins, then at least one bin has more than one item.
- Use the pigeonhole principle to find the minimum number of elements to guarantee two with a duplicate property.

- Example: How many people must be in a room to guarantee that two people have the last name begin with the same initial?

  26 alphabets, hence 27 people are must be in the room.

- Example: How many times must a single die be rolled in order to guarantee getting the same value twice?

  6 possible outcomes, hence 7 times.





# Exercise

- Example: Given the sets A, B, and C, use the addition principle, multiplication principle, and the principle of inclusion and exclusion to determine the number of elements in the following sets.

  A = {Heads ; Tails}

  B = {1; 2; 3; 4}

  C = {2; 4; 6}

  a) $|(A \times C) \cup B|$.

  b) $|(B \cup C) \cap A|$.

  c) $|A \times (B \cap C)|$.

## 14 Exercise

- Example: Among 150 college students, 83 own cars, 97 own bike, 28 own motorcycles, 53 own a car and a bike, 14 own a car and a motorcycle, 7 own a bike and a motorcycle, and 2 own all three.

  a. How many students do NOT own any of the three?

  b. How many students own a bike and nothing else?



## 15 Exercise

- Example: A computerized dating service has a list of 50 men and 50 women. Names are selected at random. How many names must be chosen to guarantee one name of each gender?

- Example: A computerized housing service has a list of 50 men and 50 women. Names are selected at random. How many names must be chosen to guarantee two names of the same gender?