Chapter 7.1 Directed Graphs

Read: 7.1

Next Class: 7.2

Introduction

- We confine our attention to (unweighted) directed graphs with no parallel arcs.
  - Two arcs from node *a* to node *b* would be parallel, but one arc from *a* to *b* and another from *b* to *a* are not parallel arcs.
- The adjacency matrix of the graph is an $n \times n$ matrix, not necessarily symmetric.
- The adjacency matrix will be a Boolean matrix, that is, a matrix whose only elements are 0s and 1s.
- There is a one-to-one correspondence between an $n \times n$ Boolean matrix and the directed graph that the matrix represents.

Directed graphs with n nodes, no parallel arcs	←– –→	$n{\times}n$ Boolean matrices

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Directed Graphs and Binary Relations

- Suppose G is a directed graph with n nodes and no parallel arcs. Let N be the set of nodes.
- If (n_i, n_j) is an ordered pair of nodes, then there either is or is not an arc in G from n_i to n_j.
- We can use this property to define a binary relation on the set N. This relation is the **adjacency relation** of the graph.

  $n_i \rho n_j \leftrightarrow$ there is an arc in G from n_i to n_j
- Conversely, if ρ is a binary relation on a set N, we can define a directed graph G with N as the set of nodes, and an arc from n_i to n_j if and only if $n_i \rho n_j$. G will have no parallel arcs.

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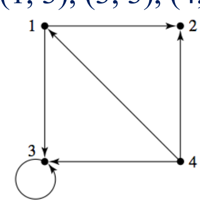
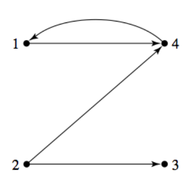
Directed Graphs and Binary Relations

- For the directed graph of the figure below, the adjacency relation is {(1, 2), (1, 3), (3, 3), (4, 1), (4, 2), (4, 3)}.



- For the set N {1, 2, 3, 4} and the binary relation {(1, 4), (2, 3), (2, 4), (4, 1)} on N, we obtain the associated directed graph:



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Directed Graphs and Binary Relations

- We have the following one-to-one correspondences:

Binary relations on n-element sets $\longleftrightarrow$ Directed graphs with n nodes, no parallel arcs

Directed graphs with n nodes, no parallel arcs $\longleftrightarrow$ $n \times n$ Boolean matrices

- If function composition is carried out on these bijections, the result is a bijection that gives us a one-to-one correspondence between binary relations and matrices.

Binary relations on n-element sets $\longleftrightarrow$ Directed graphs with n nodes, no parallel arcs $\longleftrightarrow$ $n \times n$ Boolean matrices

- Recall the reflexive, symmetric, antisymmetric, and transitive properties of a binary relation on a set.
- If a binary relation on a set N has a certain property, this will be reflected in the corresponding graph and the corresponding Boolean matrix.

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Reachability

- **DEFINITION: REACHABLE NODE** In a directed graph, node n_j is **reachable** from node n_i if there is a path from n_i to n_j.
- The "reachability" property has an interesting interpretation in each of the three equivalent forms in directed graph, adjacency relation, and adjacency matrix.
- The adjacency matrix **A** of a directed graph G with n nodes and no parallel arcs will have a 1 in position i,j if there is an arc from n_i to n_j.
- Let $\mathbf{A}^{(2)}$ be the Boolean matrix multiplication $\mathbf{A} \times \mathbf{A}$ given by:

$$\mathrm{A}^{(2)}[i, j] = \bigvee_{k=1}^{n} \left(a_{ik} \wedge a_{kj}\right)$$

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Reachability

- If a term such as $a_{i2} \wedge a_{2j}$ in this sum is 0, then either $a_{i2} = 0$ or $a_{2j} = 0$ (or both), and there is either no path of length 1 from n_i to n_2 or no path of length 1 from n_2 to n_j (or both). Thus there are no paths of length 2 from n_i to n_j passing through n_2.
- A path of length 2 from n_i to n_j will exist if and only if there is a path of length 2 passing through at least one of the nodes from 1 to n, that is, if and only if at least one of the terms in the sum is 1, $\mathbf{A}^{(2)}[i, j] = 1$.
- The matrix $\mathbf{A}^{(2)}$ indicates the presence or absence of length-2 paths, this result holds for arbitrary powers and path lengths as well.
- **THEOREM ON BOOLEAN ADJACENCY MATRICES AND REACHABILITY:** If **A** is the Boolean adjacency matrix for a directed graph G with n nodes and no parallel arcs, then $\mathbf{A}^{(m)}[i, j] = 1$ if and only if there is a path of length m from node n_i to node n_j.

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Reachability

- If node n_j is reachable from node n_i, it is by a path of some length. Such a path will be shown by a 1 as the i, j entry in **A** or $\mathbf{A}^{(2)}$ or $\mathbf{A}^{(3)}$…
- If there are n nodes in the graph, then any path with n or more arcs ($n + 1$ or more nodes) must have a repeated node. Therefore, we never need to look for a path from n_i to n_j of length greater than n.
- To determine reachability, consult element i, j in $\mathbf{A}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(n)}$.

  We can define a **reachability matrix R** by:

  $$\mathbf{R} = \mathbf{A} \vee \mathbf{A}^{(2)}, \vee \ldots \vee \mathbf{A}^{(n)}$$

n_j is reachable from n_i if and only if entry i, j in **R** is positive.

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Reachability

- If ρ is the adjacency relation for a graph G, we let **R** denote the binary relation of reachability $(n_i, n_j) \in \rho^{\mathbf{R}}$ exactly when there is a path in G from n_i to n_j.
- $\rho^{\mathbf{R}}$ is the transitive closure of ρ
- The three equivalent representations of adjacency relation, directed graph G, and adjacency matrix **A**, correspond to:

(n_i,n_j) belongs to the transitive closure of ρ $\longleftrightarrow$ n_j reachable from n_i in G $\longleftrightarrow$ R[i,j] = 1, where $R = A \vee A^{(2)} \vee \ldots \vee A^{(n)}$

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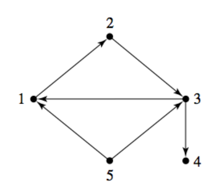
Reachability Example

- Given the directed graph, G with 5 nodes:
- The Adjacency Matrix is:



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{(4)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad \mathbf{A}^{(5)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The Boolean Sum of the matrices is:

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Reachability Example

- The adjacency relation is ρ = {(1, 2), (2, 3), (3, 1), (3, 4), (5, 1), (5, 3)}.
- Taking the transitive closure of ρ adds the following pairs: (1, 3), (2, 1), (2, 4), (3, 2), (5, 2), (5, 4), (1, 1), (1, 4), (2, 2), (3, 3).
- So we have:
  {(l, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4),
  (3, 1), (3, 2), (3, 3), (3, 4), (5, 1), (5, 2), (5, 3), (5, 4)}
- Notice that it agrees with matrix **R.**

Warshall's Algorithm

- For a graph G with n nodes, Warshall's algorithm computes a sequence of $n + 1$ matrices $\mathbf{M}_0$, $\mathbf{M}_1$, $\mathbf{M}_2$, ... , $\mathbf{M}_n$.
- For each k, $0 \leq k \leq n$, $\mathbf{M}_k[i, j] = 1$ if and only if there is a path in G from n_i to n_j whose interior nodes (i.e., nodes that are not the endpoints of the path) come only from the set of nodes $\{n_1, n_2, \dots, n_k\}$.
- Warshall's algorithm begins with $\mathbf{A} = \mathbf{M}_0$ and $\mathbf{M}_1$, $\mathbf{M}_2$,... , $\mathbf{M}_n = \mathbf{R}$ inductively.
- The base case is to let $\mathbf{M}_0 = \mathbf{A}$. Assume that $\mathbf{M}_k$ has been computed, $\mathbf{M}_{k+1}[i, j] = 1$ if and only if there is a path from n_i to n_j whose interior nodes come only from the set $\{n_1, n_2, \dots, n_{k+1}\}$.

Warshall's Algorithm

- There are two ways to see if $\mathbf{M}_{k+1}[i, j] = 1$

1. All the interior nodes come from $\{n_1, n_2, \ldots, n_k\}$, in which case $\mathbf{M}_k[i, j] = 1$. Any 1 entries in $\mathbf{M}_k$ are carried forward into $\mathbf{M}_{k+1}$.
2. Node n_{k+1} is an interior node. There must be a path from n_i to n_{k+1} whose interior nodes come from $\{n_1, n_2, \ldots, n_k\}$ and a path from n_{k+1} to n_j whose interior nodes come from $\{n_1, n_2, \ldots, n_k\}$, so $\mathbf{M}_k[i, k+1] = 1$ and $\mathbf{M}_k[k+1, j] = 1$.

Warshall's Algorithm

- **ALGORITHM Warshall**
  Warshall($n \times n$ Boolean matrix **M**)
  //Initially, **M** = adjacency matrix of a directed graph G //with no parallel arcs
    for $k = 0$ to $n - 1$ **do**
      for $i = 1$ to n **do**
        for $j = 1$ to n **do**
          $\mathbf{M}[i, j] = \mathbf{M}[i, j] \vee (\mathbf{M}[i, k+1] \wedge \mathbf{M}[k+1, j])$
        end for
      end for
    end for
  //at termination, **M** = reachability matrix of G
  endWarshall

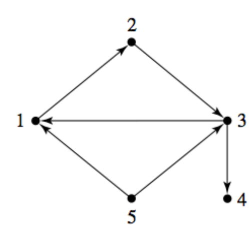
Warshall’s Algorithm

- Informally, Warshall’s algorithm works as follows:

1. Consider column $k + 1$ in $\mathbf{M}_k$.
2. For each row with a 0 entry in this column, copy that row to $\mathbf{M}_{k+1}$.
3. For each row with a 1 entry in this column, **or** that row with row $k + 1$ and write the resulting row in $\mathbf{M}_{k+1}$.

Warshall’s Algorithm Example

- Given the follow graph:
- The initial adjacency matrix is $\mathbf{M_0}$
- $\mathbf{M_4} = \mathbf{M_5} = \mathbf{M_3}$ (not shown)



$$\mathbf{M}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{M}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$