GRAPH ALGORITHMS CSE 5311

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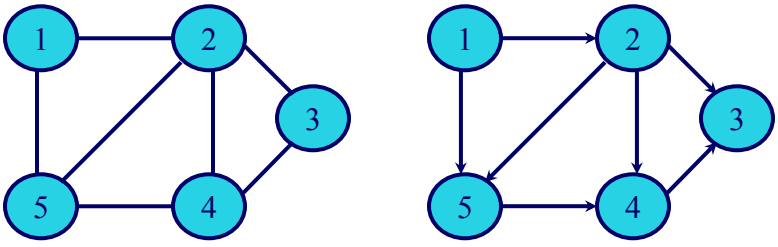
Graphs

G(V,E)

V – set of vertices or nodes
E – set of edges connecting the vertices

Edge is represented by (u,v), u,v ∈ V

Directed & Undirected



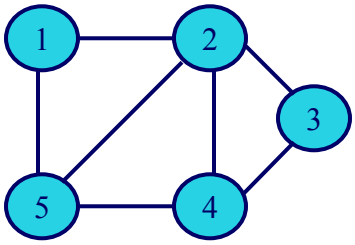
Undirected Graph Directed Graph

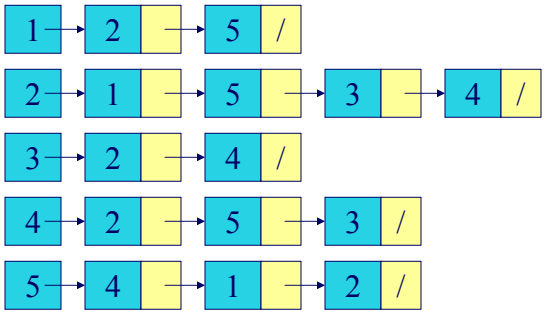
Degree	Number of edges connected to the vertex
In-degree	Number of edges coming in to the vertex
Out-degree	Number of edges going out of the vertex

Graph Representation

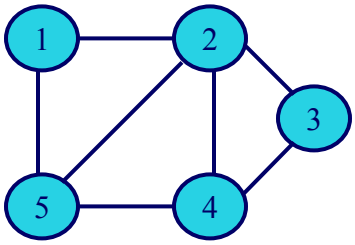
- Adjacency Lists
  - Preferred for Sparse Graphs $E << |V|^2$
  - Requires $\Theta(|V| + |E|)$ memory spaces
- Adjacency Matrix
  - Preferred for Dense Graphs $E \rightarrow |V|^2$
  - Requires $\Theta(|V|^2)$ memory spaces
  - To determine connectivity between vertices
  - Used by all-pairs shortest-paths algorithms

Adjacency List





Adjacency Matrix



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Graph Terminology

- **Path**: A path between two nodes is a sequence of intermediate nodes connected by edges
- **Cycle**: A path in a graph is a cycle if its first and last vertices are identical, no other vertex appears more than once and if it contains at least two distinct edges
- **Loop**: A loop is an edge with the start and the end nodes are the same

Graph Terminology

- **Connected Graph**: A graph in which all nodes are connected
- **Forest**: A graph that has more than one connected component
- **Acyclic** Graph: A graph that doesn't contain any cycle
- **Tree**: Connected Acyclic graph is called a tree
- **Spanning Tree**: A sub tree of a graph having all the nodes of the graph

Breadth First Search

- Simple algorithm to search a graph
- Works on both directed and undirected graphs
- Given a source, S $\in$ V, it discovers every other reachable vertex.
- It produces the breadth-first tree with shortest paths between S and other vertices

BFS

- Breadth First Search, starts at any node (Level 0) of the graph and then scans for the nearest neighbors at Level 1.
- Then it scans for the nodes at Level 2 for all nodes at Level 1
- The algorithm ends when all nodes are visited
- Uses Queue to store the visited nodes

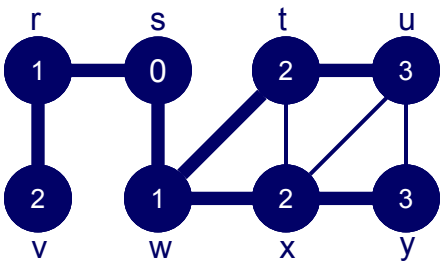
BFS: pseudo-code

```
BFS(G(V,E),s)

for each vertex u ∈ V[G] – {s}
    color[u] = white
    dist[u] = ∞
end for
color[s] = gray
dist[s] = 0
T=null
Q=null
Add(Q, s)
```

```
while Q ≠ null
    u = Delete(Q, s)
    for each v ∈ Adj_List[u]
        if color[v] = white
            color[v] = gray
            dist[v] = dist[u] + 1
            T = T U (u,v)
            Add(Q, v)
        end if
    end for
    color[u] = black
end while
```


BFS: Example



Q

T | s,w | s,r | w,t | w,x | r,v | t,u | x,y

BFS: Run Time Analysis

- Initialization takes O(|V|) time
- Adding and Deleting an element in Queue takes O(1) time. Total time for queue operations is O(|V|)
- In the iteration the adjacency list of every vertex is scanned at most once. Sum of all lengths of all adjacency lists is Θ(|E|). Total time for scanning the list is O(|E|)
- Total Running Time of BFS is O(|V| + |E|)

BFS: Applications

- Compute Connected Components of a graph
- Compute a spanning tree of a graph
- Find a cycle in a graph
- Find whether the graph is a forest
- A path between two vertices with minimum number of edges

Depth First Search

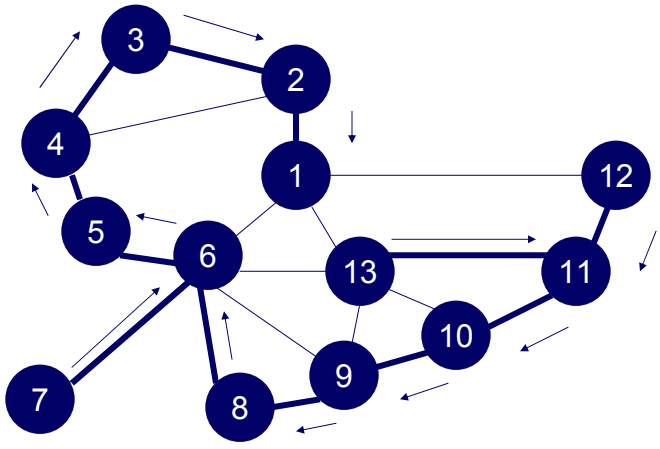
- DFS scans for the neighbors at Level 1. When it finds the first neighbor, the algorithm proceeds to scan its neighbors at Level 2.
- Goes as *deep* into the graph as possible
- The algorithm is recursive and ends when all the nodes are visited
- Uses Stack implicitly

DFS: pseudo-code

```
DFS(G(V,E))
for each vertex u ∈ V[G]
    color[u] = white
    p[u] = nil
end for
time = 0
for each vertex u ∈ V[G]
    if color[u] = white
        DFS-Visit(u)
    end if
end for
```

```
DFS-Visit(u)
color[u] = gray
time = time + 1
d[u] = time
for each v ∈ Adj[u]
    if color[v] = white
        p[v] = u
        DFS-Visit(v)
    end if
end for
color[u] = black
f[u] = time = time + 1
```

DFS: Example



DFS: Run Time Analysis

- The two loops in DFS procedure takes O(|V|) time, exclusive of time to call DFS-Visit
- DFS-Visit is called only once for each vertex
- DFS-Visit scans the Adjacency list. Sum of all lengths of all adjacency lists is Θ(|E|). Total time for scanning the list is O(|E|)
- Total Running Time of DFS is O(|V| + |E|)

DFS: Applications

- Connected Components
- Topological Sort
- Find a cycle in a graph
- Find whether the graph is a forest

DFS vs. BFS

- When a **breadth-first search** succeeds, it finds a minimum-**depth** (nearest the root) goal node

  When a **depth-first search** succeeds, the found goal node is not necessarily minimum **depth**

- For a large tree, a **depth-first search** may take an excessively long time to find even a very nearby goal node, where as **breadth-first search** can find the nearby nodes faster.

Connected Components

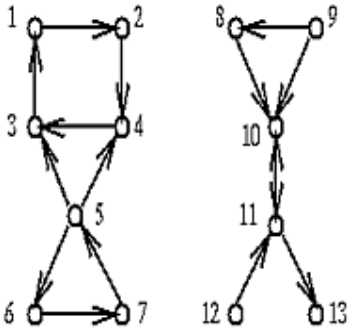
- **Problem:** Two vertices are in the same component of *G* if and only if there is some path between them. Find the number of components in a graph
- **Input :** A directed or undirected graph *G*. A start vertex *s*

Connected Components pseudo-code

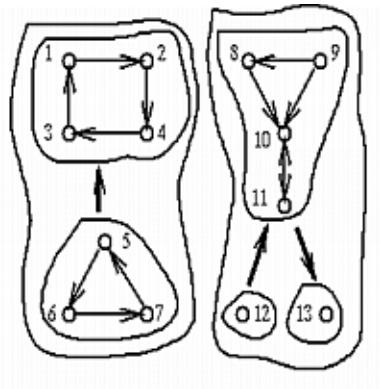
```
Procedure Connected_Components G(V,E)
Input : G (V,E)
Output : Number of Connected Components
    V' ← V;
    c ← 0;
    while  V' ≠ 0 do
          choose  u ∈ V' ;
          T ← all nodes reachable from u (by DFS_Tree)
          V' ←V' - T;
          c ← c+1;
          G_c ← T;
          T ←  0;
```

Connected Components Example

Input



Output



Topological Sort

- **Problem:** Find a linear ordering of the vertices of *V* such that for each edge *(i,j) in E*, vertex *i* is to the left of vertex *j*.
- **Input :** A directed, acyclic graph G=*(V,E)*

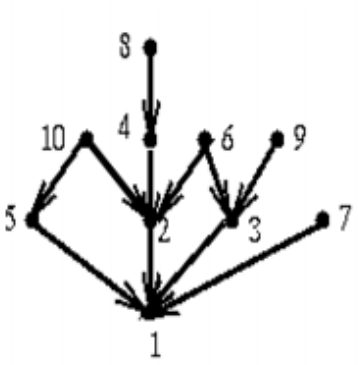
Topological Sort pseudo-code

- TopologicalSort(G)

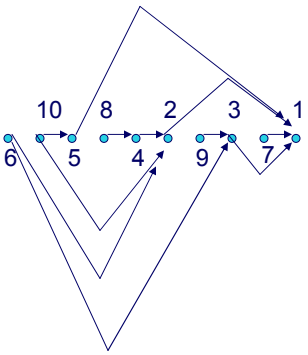
1. DFS(G)
2. As each vertex finishes, insert it on the front of a linked list
3. Return linked list of vertices

Topological Sort Example

Input



Output



Minimum Cost Spanning Tree

- Find a tree formed from graph edges that connects all the vertices in *G*, at minimum cost
- Kruskal's Algorithm
- Prim's Algorithm

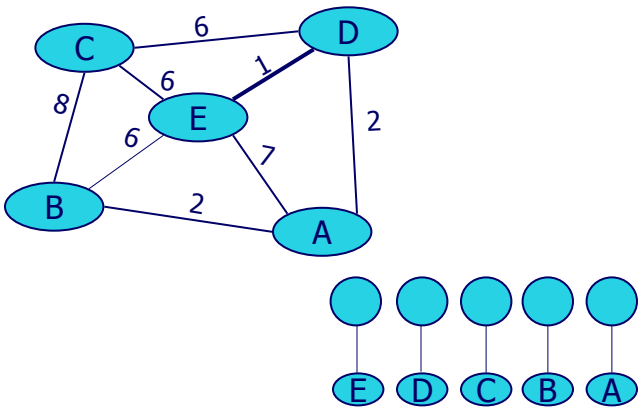
Kruskal's algorithm

- **It is a greedy algorithm which sorts the edges in the order of minimum weight and adds them to the tree, till all the nodes are added**
- **The set formed is a forest and the edges that connects the forests are also included**

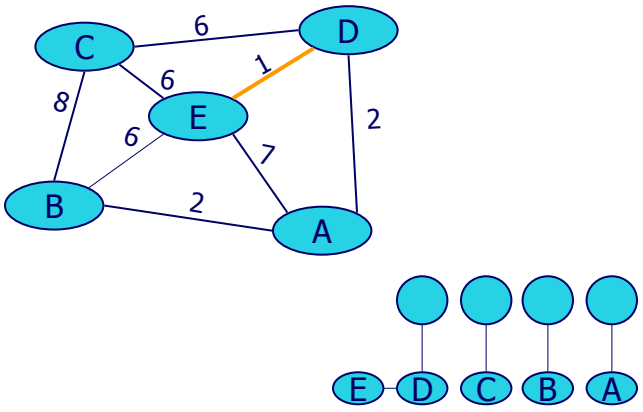
Kruskal's algorithm pseudo-code

```
Algorithm KruskalMST(G)
T ← 0;
VS ←0;
for each vertex v ∈ V do
    VS = VS ∪ {v};
sort the edges of E in non-decreasing order of weight
while |VS| > 1  do
    choose (v,w) an edge E of lowest cost;
    delete (v,w) from E;
    if v and w are in different sets W1 and W2 in VS do
          W1 = W1 ∪ W2;
          VS = VS - W2;
          T ← T∪ (v,w);
return T
```

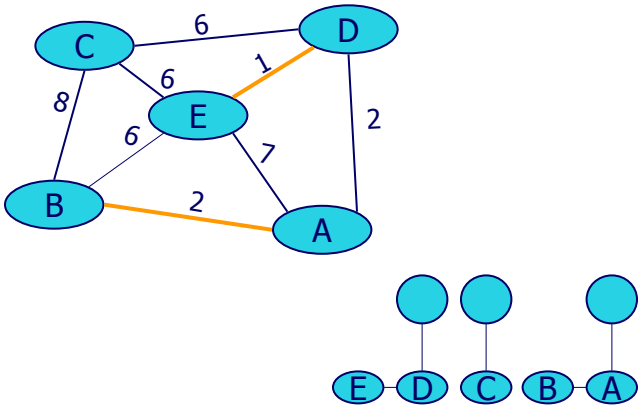

Kruskal's Algorithm Example



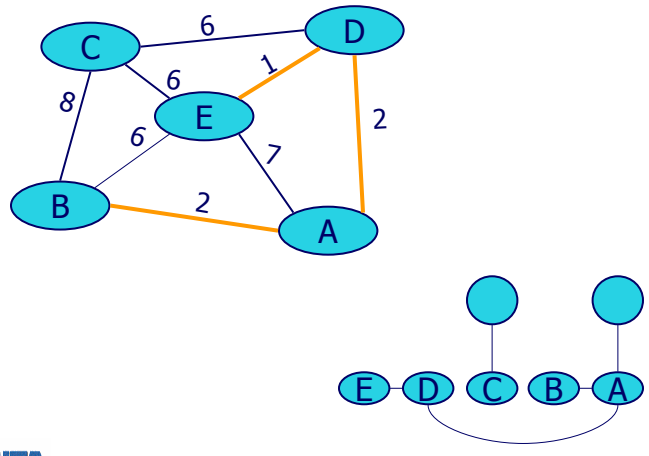
Kruskal's Algorithm Example



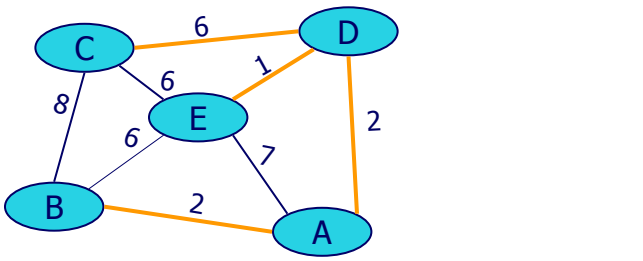
Kruskal's Algorithm Example

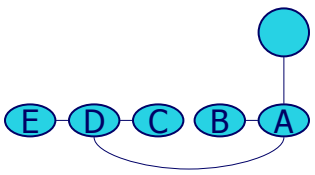


Kruskal's Algorithm Example



Kruskal's Algorithm Example





Kruskal's Algorithm Run Time Analysis

- Sorting the edges take O(E log E) time
- Adding the edges to the tree takes O(E) time
- So, the running time for the algorithm is in O(E log E)

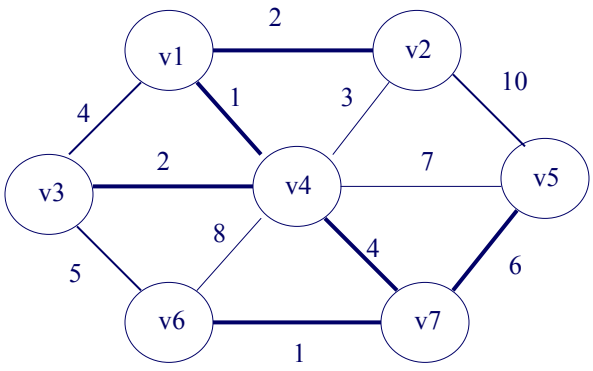
Prim's Algorithm

- Prim's algorithm for minimum spanning trees has a similar style to Dijkstra's algorithm for shortest paths
- The set formed is always a tree
- The edge with minimum weight added always connects a node that is not in the tree

Prim's algorithm pseudo-code

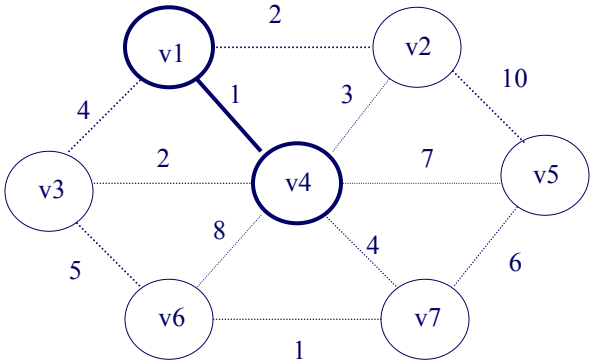
```
Algorithm PrimMST(G, r)
for each vertex v ∈ V do
    key[u] = infinity
    T= NIL
key[r] = 0
Q = V
T = NULL
while Q ≠ NULL  do
    u = Extract-Min(Q)
    for each v ∈ Adj[u]
        if v ∈ Q and w(u,v) < key[v]
                key[v] = w(u,v)
                T ← T∪ (u,v)
return T
```


Prim's Algorithm Example



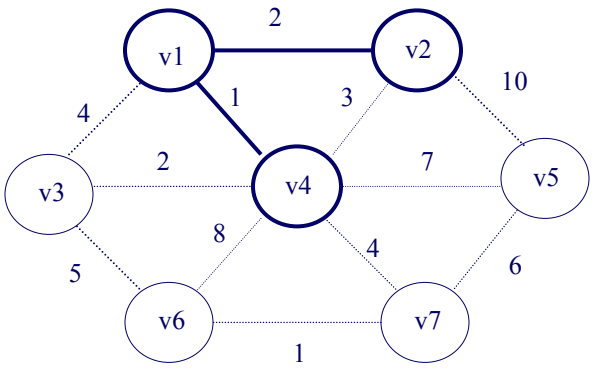
The minimum spanning tree is shown with bold edges.
The total edge cost is 16.

Prim's Algorithm Example



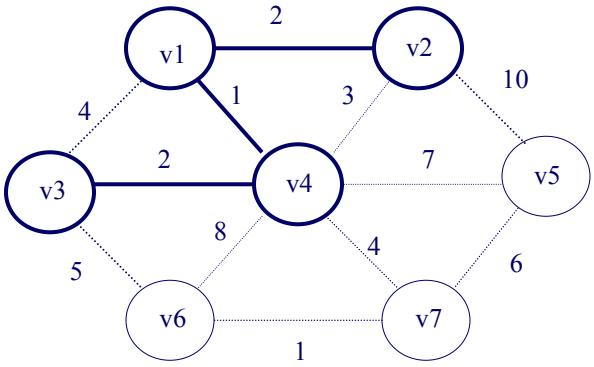
We start with v1, and place this into the tree. The rest of the vertices are not in the tree. The cheapest edge we can add from v1 to the other vertices is (v1, v4).

Prim's Algorithm Example



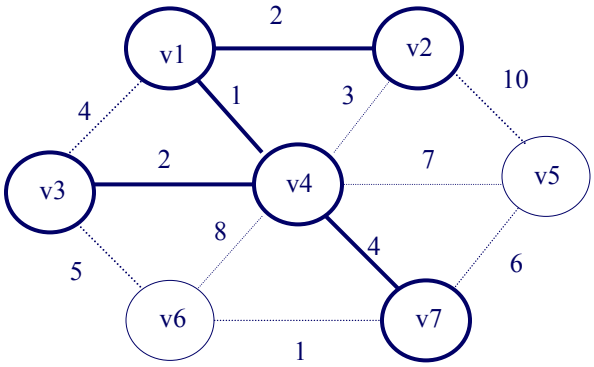
Now v1 and v4 are in the tree. The cheapest edge we can add is either (v1, v2) or (v4, v3). We choose (v1, v2).

Prim's Algorithm Example



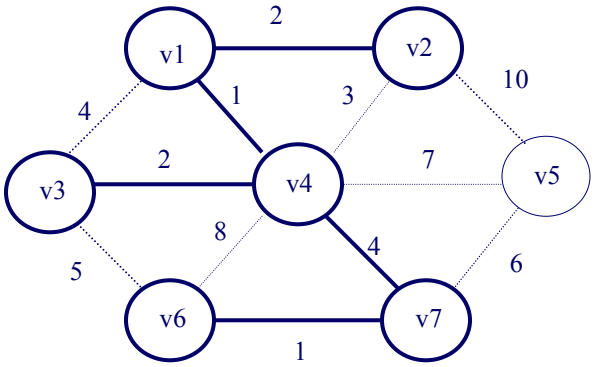
Now v1, v2, and v4 are in the tree. The cheapest edge we can add is (v4, v3).

Prim's Algorithm Example



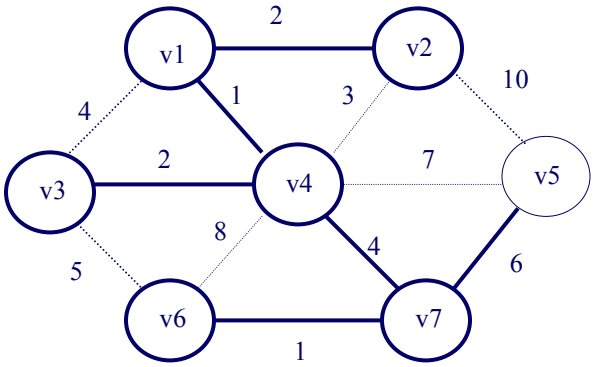
Now v1, v2, v3, and v4 are in the tree. The cheapest edge we can add is (v4, v7).

Prim's Algorithm Example



Now v1, v2, v3, v4, and v7 are in the tree. The cheapest edge we can add is (v7, v6).

Prim's Algorithm Example



Now v1, v2, v3, v4, v6 and v7 are in the tree. The cheapest edge we can add is (v7, v5). We are now done. This is optimal.

Prim's Algorithm Run Time analysis

- The initialization takes O(V) time
- The Extract-Min takes O(log V) time if it is implemented as binary heap
- For V iterations Extract-Min takes O(VlogV) time
- The key assignment takes O(log V) time
- For all edges the key assignment takes O(E log V) time.
- Hence the total time is O(VlogV + ElogV) = O(ElogV). This is same as Kruskal's algorithm

MCST: Applications

- Used in Networking protocols
  - Multicast Routing

Single Source Shortest Path

- Given graph (directed or undirected) G = (V,E) with weight function w: E → R and a vertex s, find for all vertices in V the minimum possible weight for path from s to v.

- BellMan-Ford's Algorithm
- Dijkstra's Algorithm

Relaxing an Edge

- Relaxing an edge consists of testing whether we can improve the shortest path to v by going through u
- If there is a shortest path then update Distance[v] and Predecessor[v]=u

Relax(u,v)

```
If d[v] > d[u] + w(u,v)
        d[v] = d[u] + w(u,v)
        p[v] = u
```

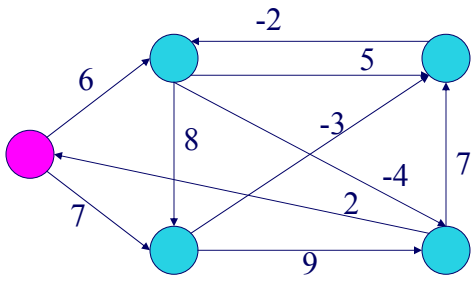
Bellman-Ford's Algorithm

- The algorithm uses relaxation
- The d[v], the weight of the path from source s to destination v, is relaxed progressively until it reaches the minimum weight
- The algorithm returns true only if the graph does not contain any negative cycles

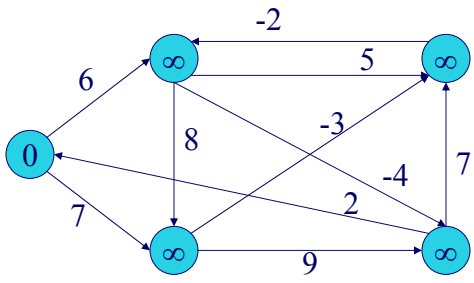
Bellman-Ford's Algorithm pseudo-code

```
BellmanFord(graph (G,w), vertex s)
    for each vertex v ∈ V
        d[v] = infinity
        p[v] = NIL
    d[s] = 0
    for i ← 1 to |V| - 1 do
        for (u,v) ∈ E[G] do
            Relax(u,v)
    for (u,v) ∈ E[G] do
        if d[v] > d[u] + w(u,v) then
            return false
    return true
```

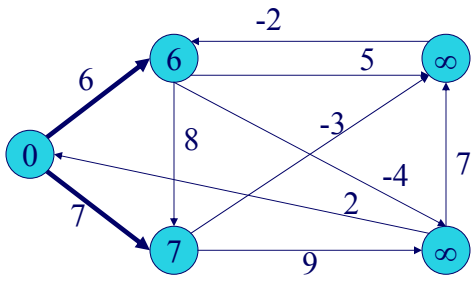
Bellman-Ford's Algorithm Example



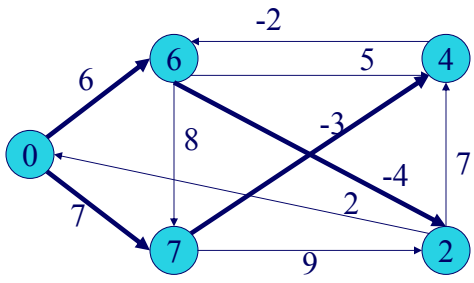
Bellman-Ford's Algorithm Example



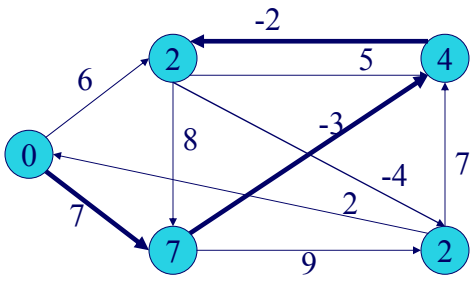
Bellman-Ford's Algorithm Example



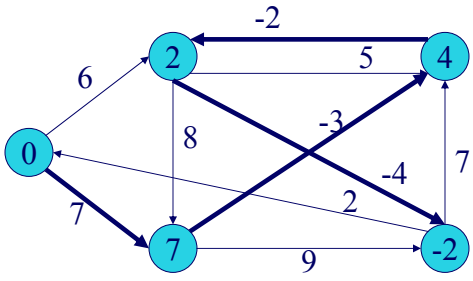
Bellman-Ford's Algorithm Example



Bellman-Ford's Algorithm Example



Bellman-Ford's Algorithm Example



Bellman-Ford's Algorithm Run Time Analysis

- The Bellman-Ford algorithm computes the distance from source s to all other vertices of G
- Determine that G contains a negative-weight cycle,
- In O(VE) time

Dijkstra's Algorithm

- Faster than Bellman-Ford's Algorithm
- Dijkstra's algorithm is based on the greedy method
- It starts with the source and builds the spanning tree by adding one edge at a time which gives the shortest path from the source to the vertex not in the spanning tree

Dijkstras Algorithm pseudo-code

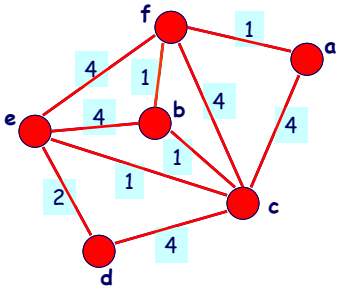
Dijkstra(G,s)

```
for each vertex v ∈ V
   d[v] = infinity
   p[v] = NIL
   d[s] = 0
S = NULL
Q = V
while Q ≠ NULL  do
   u = Extract-Min(Q)
   S ← S∪ {u}
   for each v ∈ Adj[u]
         Relax(u,v)
```


Dijkistra's Algorithm

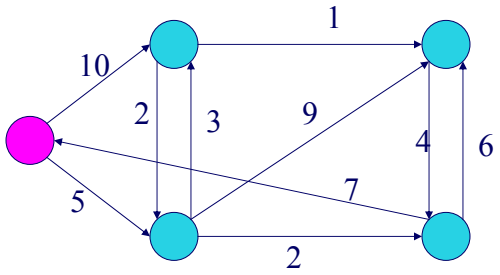
- It assumes the path length does not increase when more edges are added to the path. If a node with a negative incident edge were to be added late to the vertex list for which decisions have been made, it could mess up distances for vertices already in the list.
- This is almost identical to that of Prim's algorithm. The only difference here is that the Dijkstra's algorithm stores the distance from the source to current vertex, while Prim's algorithm stores the cost of the minimum-cost edge connecting a vertex in V to *u*.

Dijkstra's Algorithm Example

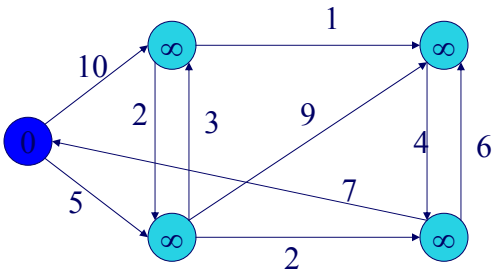


D [a	b	c	d	e	f]	C
x	x	x	0	x	x	ϕ
x	x	4	0	2	x	{d}
x	6	3	0	2	6	{d,e}
7	4	3	0	2	6	{c,d,e}
7	4	3	0	2	5	{b,c,d,e}
6	4	3	0	2	5	{b,c,d,e,f}
6	4	3	0	2	5	{a,b,c,d,e,f}

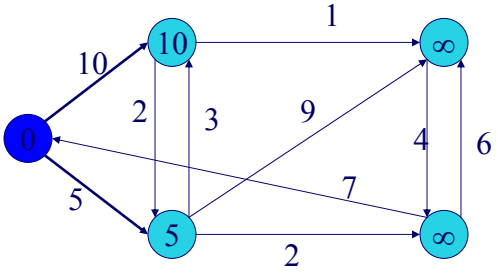
Dijkstra's Algorithm Example



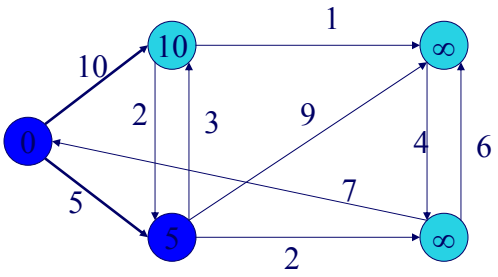
Dijkstra's Algorithm Example



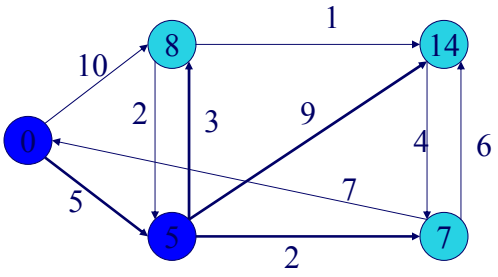
Dijkstra's Algorithm Example



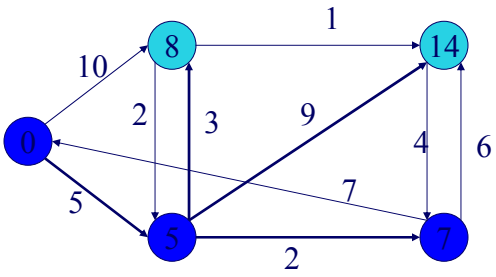
Dijkstra's Algorithm Example



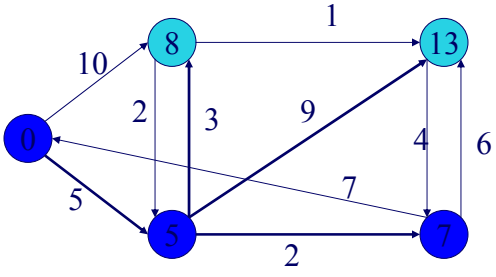
Dijkstra's Algorithm Example



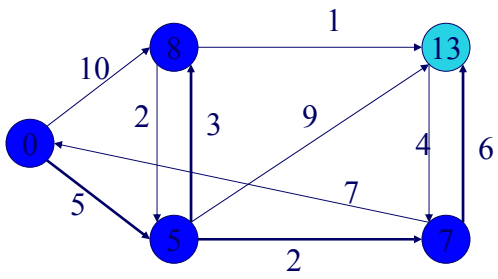
Dijkstra's Algorithm Example



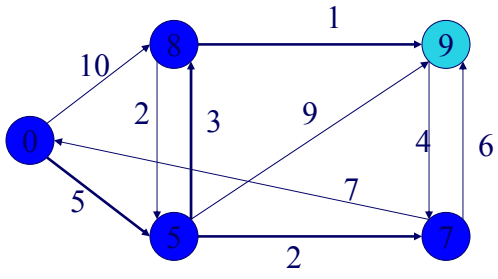
Dijkstra's Algorithm Example



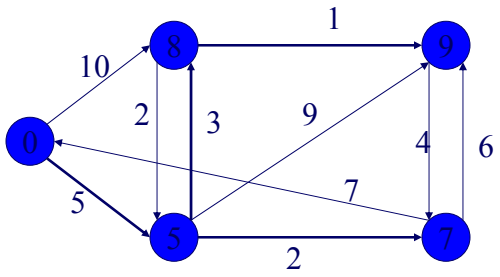
Dijkstra's Algorithm Example



Dijkstra's Algorithm Example



Dijkstra's Algorithm Example



Dijkstra's Algorithm Run Time Analysis

- The initialisation takes O(V) time
- There are V iterations of Extract-Min
- There are E iterations of distance assignment
- Time = V x $Time_{Extract\text{-}Min}$ + E x $Time_{dist\ assign}$

Q	$T_{EXTARCT\text{-}MIN}$	$T_{DIST\text{-}ASSIGN}$	Total Time
Array	O(V)	1	$O(V^2)$
Binary Heap	O(log V)	O(log V)	O(E logV)
Fibonacci Heap	O(log V)	1	O(E + V log V)

SSSP: Applications

- Used in Routing Protocols
  - Dijkstra's Algorithm is used in Link State Routing
  - Bellman-Ford's Algorithm is used in Distance Vector Routing

All Pairs Shortest Path

The all-pairs-shortest-path problem is generalization of the single-source-shortest-path problem for all nodes.

Problem is to find the shortest paths between all pairs of vertices $V_i, V_j \in V$ such that $i \neq j$.

Floyd's Algorithm

- $d_{s,t}^{(i)}$ – the shortest path from s to t containing only vertices v_1, ..., v_i
- $d_{s,t}^{(0)} = w(s,t)$

- $d_{s,t}^{(k)} = w(s,t)$ if k = 0

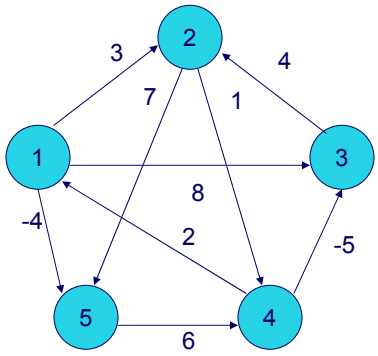
  $\min\{d_{s,t}^{(k-1)}, d_{s,k}^{(k-1)} + d_{k,t}^{(k-1)}\}$ if k > 0

Floyd's Algorithm pseudo-code

```
FloydWarshall (matrix W, integer n)
  D(0) ← W
  for k ← 1 to n do
     for i ← 1 to n do
        for j ← 1 to n do
           d_ij^(k) ← min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))
  return D(n)
```

Floyd's Algorithm Example

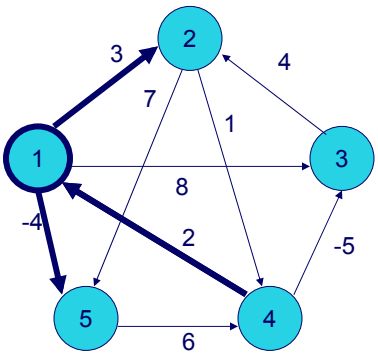
Adjacency Matrix



	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

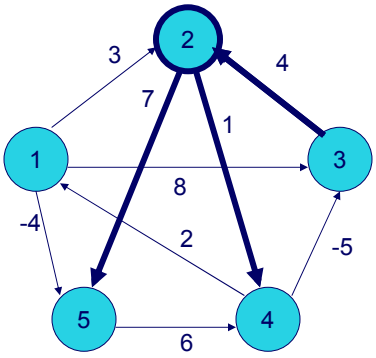
Floyd's Algorithm Example

Shortest Path Matrix $D^{(1)}$



	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

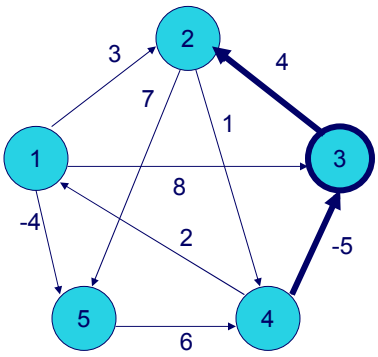
Floyd's Algorithm Example



Shortest Path Matrix $D^{(2)}$

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

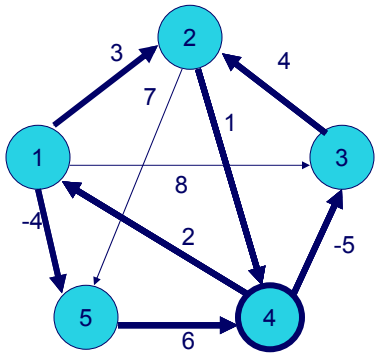
Floyd's Algorithm Example



Shortest Path Matrix $D^{(3)}$

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	-1	-5	0	-2
5	∞	∞	∞	6	0

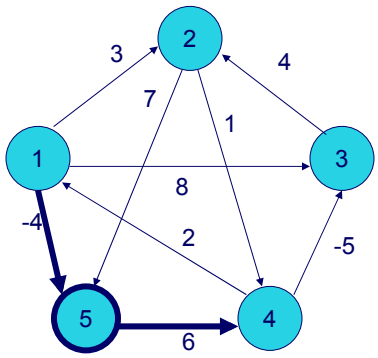
Floyd's Algorithm Example



Shortest Path Matrix $D^{(4)}$

	1	2	3	4	5
1	0	3	-1	4	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

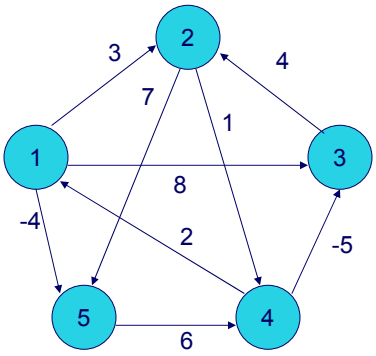
Floyd's Algorithm Example



Shortest Path Matrix $D^{(5)}$

	1	2	3	4	5
1	0	3	-1	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

Floyd's Algorithm Example



Shortest Path Matrix $D^{(5)}$

	1	2	3	4	5
1	0	3	-1	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

Floyd's Algorithm Run Time Analysis

- Running Time of $O(|V|^3)$
- Negatively weighed edges may be present.
- Negatively weighted cycles cause problems with the algorithm.

Negative Cycle Detection

- Execute Floyd's Algorithm. If at least one diagonal entry contain a negative value then there is a negative cycle
  - Complexity is $O(V^3)$
- Execute Bellman-Ford Algorithm. If it returns false then the graph contain a negative cycle
  - Complexity is $O(VE)$

Faster Algorithm for All Source Shortest Path

- Try using the existing Single Source Shortest Path for all sources
- If Bellman-Ford's algorithm is again to find the shortest paths for all sources then the complexity is $O(V^2E) = O(V^4)$
- Bellman-Ford's algorithm is worse than Floyd's algorithm

Faster Algorithm for All Source Shortest Path

- Use Dijkstra's Algorithm to compute shortest paths for all sources
- Runs in time O(V (E log V)) with binary heap and O(V*(E + V log V) with Fibonacci heaps
- But cannot have negative edges

Reweighting the edges

- A reweighting function ŵ should satisfy two properties
  - For all pairs of vertices u, u ∈ V, a path p is a shortest path from u to v using the weight function w if and only if p is also a shortest path from u to v using weight function ŵ
  - For all edges (u,v) the new weight ŵ(u,v) is non-negative

Reweighting the edges

- For a weighted, directed graph G=(V,E) with weight function
  - w : E → R (weight of edges)
  - h : V → R (distance vector calculated using Bellman-Ford's algorithm)
- $\hat{w}(u,v) = w(u,v) + (h(u) – h(v))$

Johnson's Algorithm

1. Add a node S to the graph with zero weight edges connecting to all nodes of the graph
   - It is done to calculate the weighting function using Bellman-Ford's algorithm. If any one node does not have path to all other nodes then the weight of some nodes will be infinite

Johnson's Algorithm

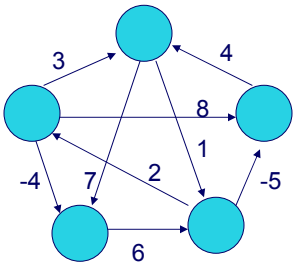
2. Execute Bellman-Ford's algorithm on the new graph
3. If it returns false then there is a negative cycle in the graph and hence abort the algorithm
4. Reweigh all the edges

Johnson's algorithm

5. For all nodes execute Dijkstra's algorithm to find the shortest path to all other nodes
6. Recalculate the correct shortest-path weight

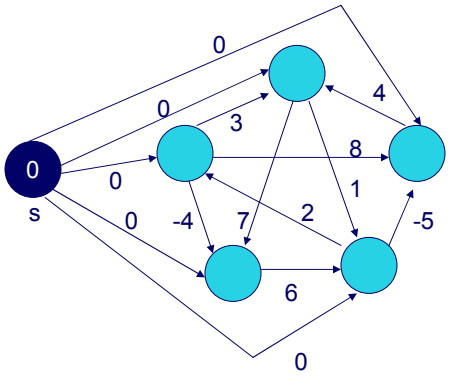
$$dist(u)(v) = \hat{g}(u,v) - (h(v) - h(u))$$

Johnson's algorithm Example



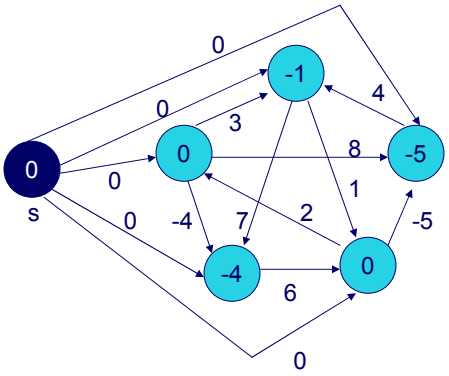
Original Graph

Johnson's algorithm Example



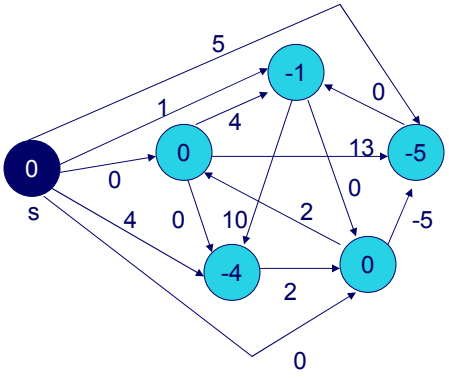
Graph with the 's' node added with zero weight edges

Johnson's algorithm Example



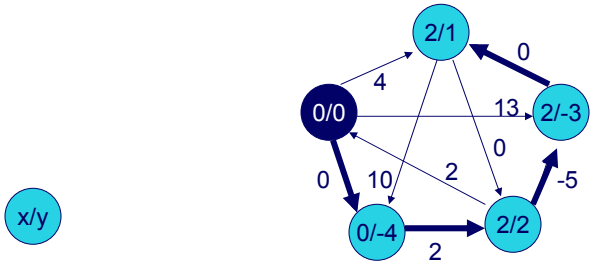
Weight function calculated using Bellman Ford's algorithm

Johnson's algorithm Example



Reweighted non-negative edges

Johnson's algorithm Example



x – weight using Dijkstra's algorithm

y – actual recalculated weight

Reweighted non-negative edges

Johnson's algorithm Run Time analysis

- Find the weighting function h such that $w(u, v) >= 0$ for all *edges* by using Bellman-Ford, or determine that a negative-weight cycle exists.
  - Time = $O(VE)$.
- Run Dijkstra's algorithm from each vertex using w.
  - Time = $O(VE \log V)$. (when Fibonacci heap is used)
- Reweight each shortest-path length $w(p)$ to produce the shortest-path lengths $w(p)$ of the original graph.
  - Time = $O(V^2)$.
- Total time = $O(VE \log V)$. Better for sparse graphs

Representation of Graph Files

- GraphWiz is a set of graph drawing tools developed by the AT&T Research Labs
- Two types of file format
  - .dot for directed graphs
  - .neato for undirected graphs
- GraphWiz and WinGraphViz are open source software. Contains both executables and libraries

.dot file

```
/* Nodes = 4 */
/* Edges = 5 */
/* Directed = 1 *//**Random**/
/* Min In-Degree = 2 */
/* Max In-Degree = 4 */
/* Min Out-Degree = 2 */
/* Max Out-Degree = 4 */
/* Weighted = 1 */
/* Min Weight = 3 */
/* Max Weight = 10 */
```

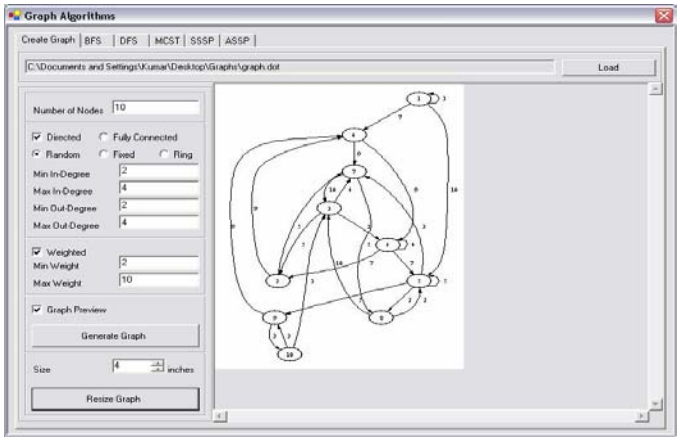
```
digraph G {
    size = "7,7";
1 -> 3 [label="6"];
2 -> 3 [label="3"];
2 -> 2 [label="5"];
3 -> 4 [label="7"];
3 -> 3 [label="3"];
}
```

.neato file

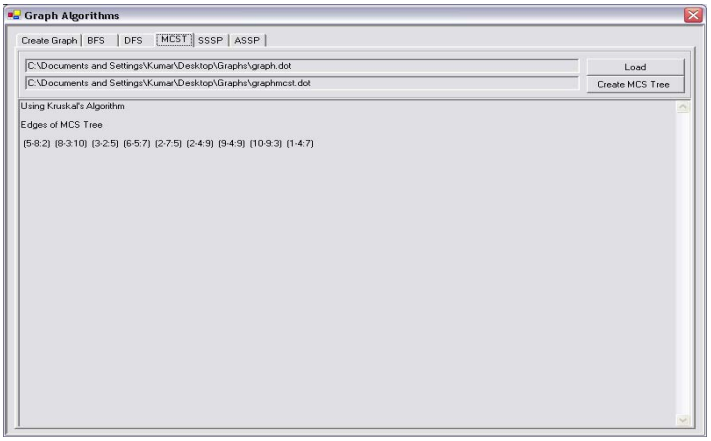
```
/* Nodes = 4 */
/* Edges = 5 */
/* Directed = 0 *//**Random**/
/* Min Degree = 2 */
/* Max Degree = 4 */
/* Weighted = 1 */
/* Min Weight = 3 */
/* Max Weight = 10 */
```

```
graph G {
    size = "7,7";
1 -- 3 [label="6"];
2 -- 3 [label="3"];
2 -- 2 [label="5"];
3 -- 4 [label="7"];
3 -- 3 [label="3"];
}
```

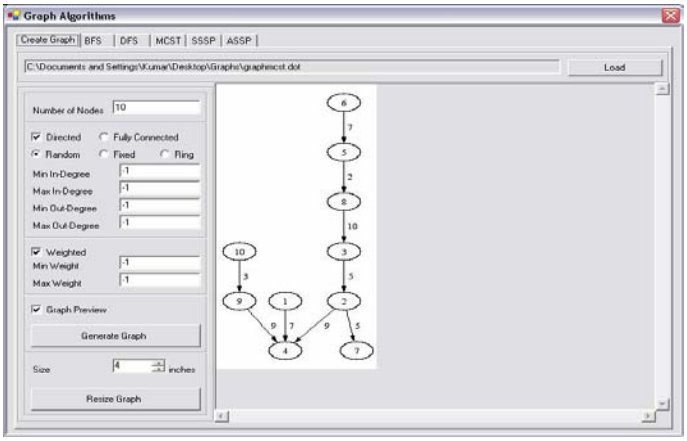

Graphs Algorithms Implementation



Graph Algorithms Implementation



Graph Algorithms Implementation



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- *Robert Sedgewick,* **Algorithms in Java**, 3rd Edition, Addison Wesley Professional. The sample chapter for **Shortest Paths Graph Algorithms** is available at http://www.informit.com/articles/article.asp?p=169575
- *Donald B.Johnson*, The Pennsylvania state university, **Efficient Algorithms for shortest path in Sparse Networks**
- **WinGraphViz** http://www.research.att.com/sw/tools/graphviz
- **All source shortest paths** - http://compgeom.cs.uiuc.edu/~jeffe/teaching/373/notes/12-apsp.pdf