Flow Networks

Topics

Flow Networks
Residual networks
Ford-Fulkerson’s algorithm
Ford-Fulkerson's Max-flow Min-cut Algorithm

Chapter 7
Algorithm Design ***Kleinberg and Tardos***

Flow Networks

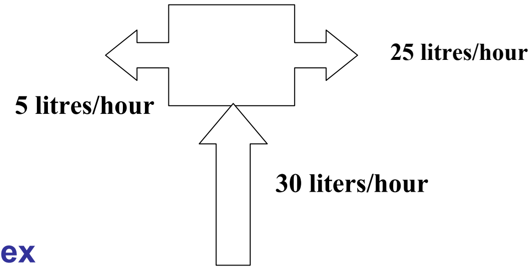
A directed graph can be interpreted as a flow network to analyse material flows through networks.

Material courses through a system from a source (where it is produced) to a sink (where it is consumed).
Examples :
- **Water through pipelines**
- **Newspapers through distribution system**
- **Electricity through cables**
- **Cars on a production line**
  - **on roads**

The source produces the material at a steady rate .
The sink consumes the material at a steady rate

Flow: the rate at which the material moves from one point to another

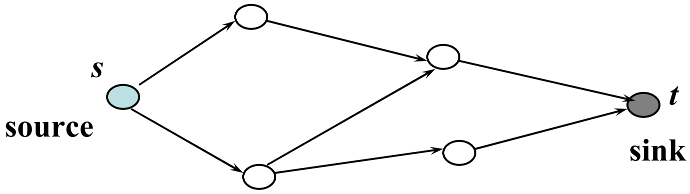
100 litres of water per hour in a pipe
30 Amperes of electric current in a circuit



The rate at which a material enters a vertex = the rate at which the material leaves the vertex

---

The flow network $G = (V,E)$ is a directed graph in which each edge $(u,v) \in E$ has a nonnegative capacity $c(u,v) \geq 0$.
If $(u,v) \notin E$ then $c(u,v) = 0$.
A flow network has a source vertex s, and a sink vertex t.
For every vertex $v \in V$ there is a path from s to v and v to t in a connected graph.



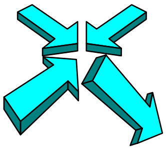
A **flow** in G is a real-valued function $f : V \times V \rightarrow R$ that satisfies the following three properties:

1. **Capacity constraint** : For all $u,v \in V$, we require $f(u,v) \leq c(u,v)$.
The net flow from one vertex to another must not exceed the given capacity.

2. **Skew symmetry** : For all $u,v \in V$, we require $f(u,v) = -f(v,u)$.

The net flow from a vertex u to a vertex v is the negative of the net flow in the reverse direction.
The net flow from a vertex to itself is zero for all $u \in V$, that is $f(u,u) = 0$.

3. **Flow conservation** : For all $u \in V - \{s,t\}$, we require



$$\sum_{v \in V} f(u,v) = 0$$

The total net flow out of a vertex other than the source or sink is zero.

The quantity $f(u,v)$ can be negative or positive, it is called the net flow from vertex u to v.

The value of a flow is defined as

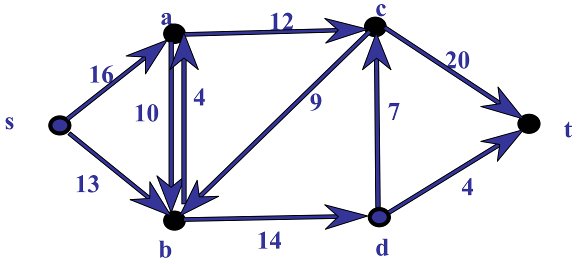
$$|f| = \sum_{v \in V} f(s,v)$$

In the maximum-flow problem, we are given a flow network G with source s and sink t, and we wish to find a flow of maximum value from s to t.

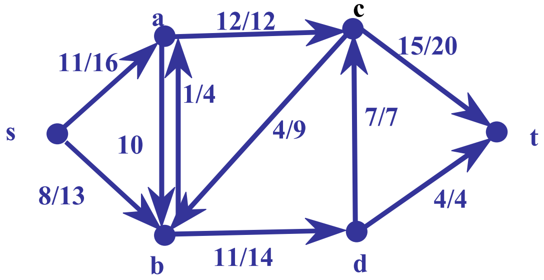
There is no net flow between u and v if there is no edge between them.
If $(u,v) \notin E$ and $(v,u) \notin E$, then $c(u,v) = c(v,u) = 0$.
Hence, the capacity constraint, $f(u,v) \leq 0$ and $f(v,u) \leq 0$.
By skew symmetry, $f(u,v) = -f(v,u)$,
therefore, $f(u,v) + f(v,u) = 0$.

Nonzero net flow from vertex u to vertex v implies that $(u,v) \in E$ or $(v,u) \in E$ (or both).

Consider the network *G=(V,E)* shown in the figure below. The network is for a transport system that transports crates of an item from source vertex *s* to sink vertex *t* through a number of intermediate points. Each edge *(u,v)* $\in$ *E* in the network is labeled with its capacity *c(u,v).*



Let us consider a flow in *G*, $|f|$=29
If *f(u,v)* >0, edge (*u,v*) is labeled *f(u,v)/c(u,v)*
If *f(u,v)* $\leq$ 0, the edge is labeled by its capacity only.



The positive net flow entering a vertex *v* is defined by

$$\sum_{\substack{u\in V\\ f(u,v)>0}} f(u,v)$$

Initially, c (a ,b) = 8, and c (b, a) = 3 -- Fig. a.
f (a, b) = 5 and f (b, a) = 2, -- Fig. b
the net flow is shown as 3/8 in direction a to b – Fig. c

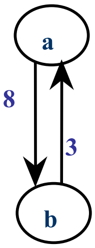


Fig.a

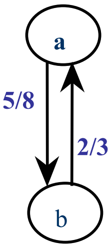


Fig.b

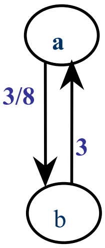


Fig.c

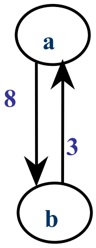


Fig.a

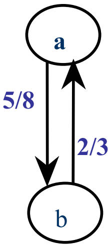


Fig.b

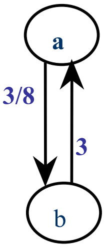


Fig.c

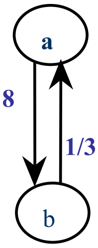


Fig.d

**If we increase the flow from
b to a from 2 to 6 then the net flow is 1/3 in the direction
b to a as shown in Fig. d.**

The Ford Fulkerson method

The method is iterative,
Starts with $f(u,v)$ for $(u,v) \in V$, initial flow of value 0.
The method is based on the augmenting path which is defined as a path from *s* to *t* along which we can push more flow and then augment flow along this path.

Procedure Ford_Fulkerson_method(*G,s,t*)

```
1. f ← 0;
2. while there exists an augmenting path p
3.        do augment flow along path p
4. return f
```

Residual Networks

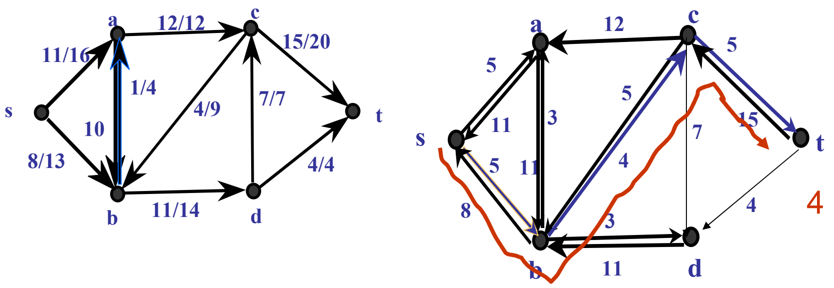
Consider a flow network $G(V,E)$ with source s and sink t and let *f* be a flow in *G*.
Consider a pair of vertices $u,v \in V$.
Residual capacity between u and v is given by

$$r(u,v) = c(u,v) - f(u,v)$$

- the additional net flow we can push from u to v before exceeding the capacity.
  For example, if $c(u,v) = 25$ and $f(u,v) = 19$, then $r(u,v) = 6$.

If $f(u,v) < 0$ then $r(u,v) > c(u,v)$

Given a flow network $G=(V,E)$ and a flow *f*, the residual network of *G* induced by *f* is $G_f=(V,E_f)$,
where $E_f = \{(u,v) \in V \times V : r(u,v) > 0\}$



Each edge in the residual network can admit positive net flow only.

The residual network may include several edges that are not in the original network, $(u,v) \in E_f$ and $(u,v) \notin E$ is possible (E_f is not a subset of E). However, (u,v) appears in G_f only if $(v,u) \in E$ and there is a positive flow from v to u. Because the net flow $f(u,v)$ is negative,

$$r(u,v) = c(u,v)-f(u,v) > 0 \text{ and } (u,v) \in E_f$$

An edge (u,v) can appear in a residual network only if at least one of (u,v) and (v,u) appears in the original network.

$$|E_f| \leq 2|E|$$

Augmenting Paths

It is a simple path from s to t in G_f. Each edge (u,v) on an augmenting path admits some additional positive net flow from u to v without violating the capacity constraint on the edge. The residual capacity of a path p is given by,

$$r(p) = \min \{ r(u,v) : (u,v) \text{ is in } p \}$$

Let's define a flow function f_p,

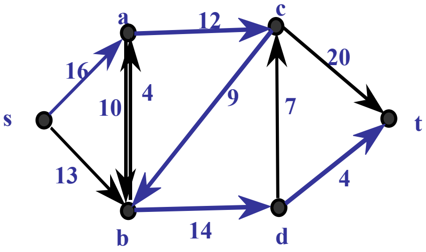
$$f_p = \begin{cases} r(p) \ \ if \ \ (u,v) \ \ is \ \ on \ \ p, \\ -r(p) \ \ if \ \ (v,u) \ \ is \ \ on \ \ p \\ 0 \ \ otherwise \end{cases}$$

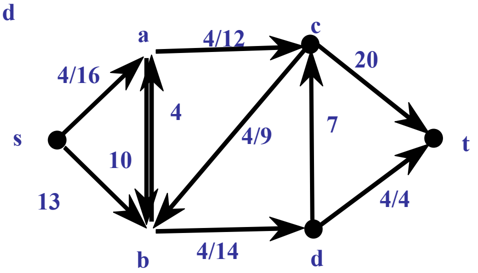
f_p is a flow in G_f with value $|f_p| = r(p) > 0$.
If we add f_p to f, we get another flow in G whose value is closer to the maximum.

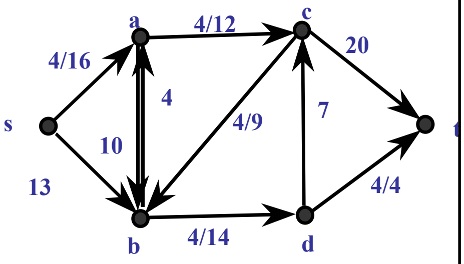
Algorithm

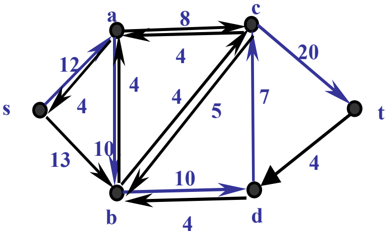
Procedure Ford-Fulkerson(G,s,t)
Input : Flow Network $G(V,E)$
Output : Maximum flow for the given network

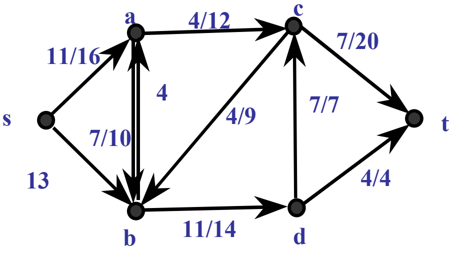
```
for each edge $(u,v) \in E$
    do    $f[u,v] \leftarrow 0$;
          $f[v,u] \leftarrow 0$;
while there exists a path p from $s$ to $t$ in the
                              residual network $G_f$
    do    $r(p) \leftarrow \min\{r(u,v) : (u,v)$ is in $p\}$;
          for each edge $(u,v)$ in $p$
              do    $f[v,u] \leftarrow - f[u,v]$;
                    $f[u,v] \leftarrow f[u,v] + r(p)$;
return
```

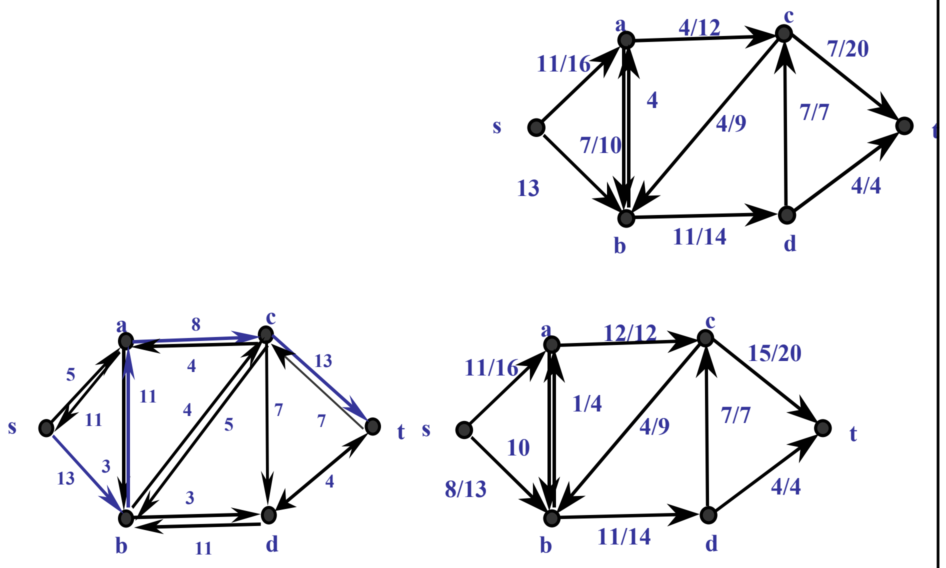

a
12
c
16
20
s
10
4
9
7
t
13
4
b
14
d

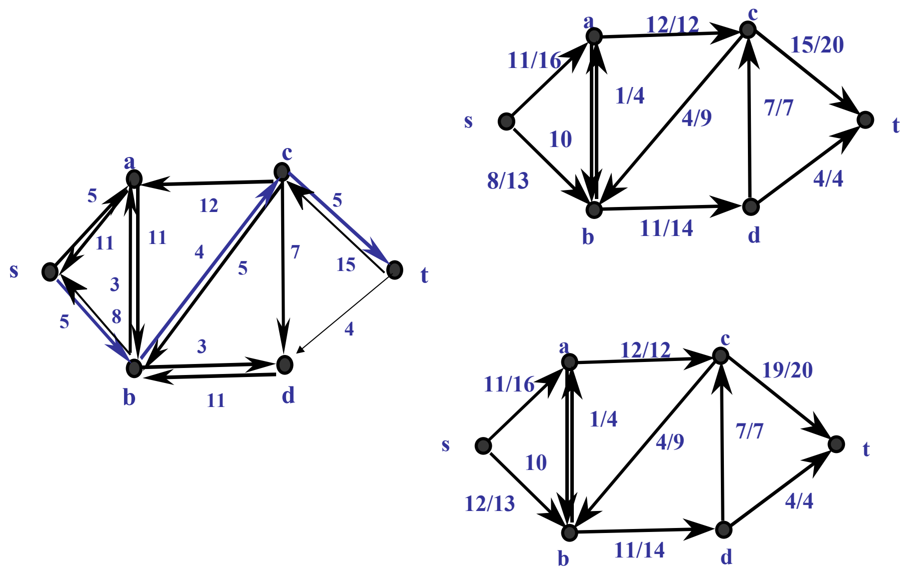

a
4/12
c
4/16
20
4
4/9
7
s
t
10
13
4/4
b
4/14
d

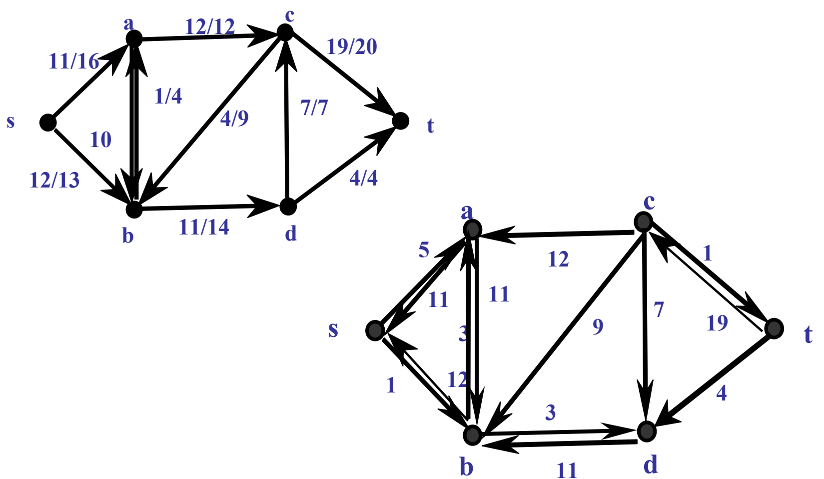

a
4/12
c
4/16
20
4
4/9
7
s
10
13
4/4
b
4/14
d


a
8
c
12
4
20
4
s
4
4
5
7
t
13
10
4
10
b
4
d


a
4/12
c
11/16
7/20
4
s
4/9
7/7
t
7/10
13
4/4
b
11/14
d


a
4/12
c
11/16
7/20
4
4/9
7/7
s
7/10
t
13
4/4
b
11/14
d
a
8
c
5
4
13
11
4
5
7
7
s
11
t
13
3
3
4
b
11
d
a
12/12
c
11/16
15/20
1/4
4/9
7/7
s
10
t
8/13
4/4
b
11/14
d

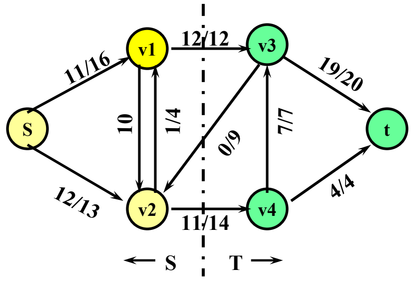

a
12/12
c
11/16
15/20
1/4
4/9
7/7
s
10
t
8/13
4/4
b
11/14
d
a
c
5
12
5
11
11
4
5
7
15
s
t
3
8
5
3
4
b
11
d
a
12/12
c
11/16
19/20
1/4
4/9
7/7
s
10
t
12/13
4/4
b
11/14
d



Ford Fulkerson – cuts of flow networks

New notion: cut (S,T) of a flow network

A cut (S,T) of a flow network G=(V,E) is a partiton of V into S and T = V \ S such that s ∈ S and t ∈ T.



In the example:

S = {s,v1,v2) , T = {v3,v4,t}

Net flow f(S ,T) = f(v1,v3) + f(v2,v4) + f(v2,v3)

= 12 + 11 + (-0) = 23

Capacity c(S,T) = c(v1,v3) + c(v2,v4)

= 12 + 14 = 26

Implicit summation notation: $f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v)$

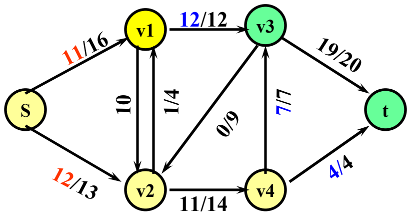
Cuts of Flow slides prepared by Shwetha and Pradeep

Ford Fulkerson – cuts of flow networks

Lemma:

the value of a flow in a network is the net flow across any cut of the network

$$\boxed{f(S, T) = |f|}$$



▶ proof

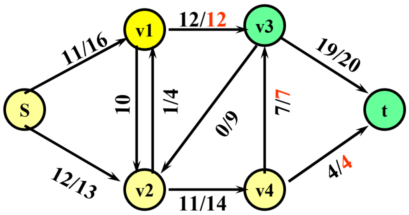
Ford Fulkerson – cuts of flow networks

Assumption:

The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G

Lemma: $\boxed{|f| \leq c(S, T)}$

$$\begin{aligned} |f| &= f(S, T) \\ &= \sum_{u \in S} \sum_{v \in T} f(u, v) \\ &\leq \sum_{u \in S} \sum_{v \in T} c(u, v) \\ &= c(S, T) \end{aligned}$$



F. Fulkerson: Max-flow min-cut theorem

If f is a flow in a flow network G = (V,E) with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. | f | = c (S, T) for some cut (S, T) of G.

proof:

(1) $\Rightarrow$ (2):

We assume for the sake of contradiction that f is a maximum flow in G but that there still exists an augmenting path p in G_f.

Then as we know from above, we can augment the flow in G according to the formula: $f' = f + f_p$. That would create a flow f' that is strictly greater than the former flow f which is in contradiction to our assumption that f is a maximum flow.

F. Fulkerson: Max-flow min-cut theorem

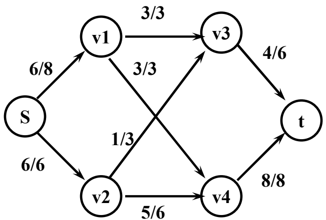
If f is a flow in a flow network G = (V,E) with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. | f | = c (S, T) for some cut (S, T) of G.

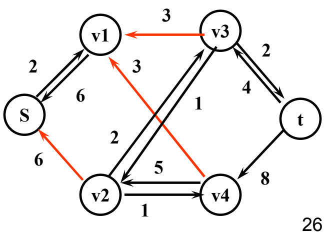
proof:

(2) $\Rightarrow$ (3):

original flow network G



residual network G_f



F. Fulkerson: Max-flow min-cut theorem

If f is a flow in a flow network $G = (V,E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

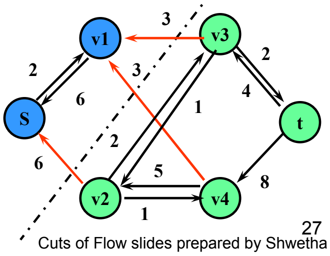
proof:

$(2) \Rightarrow (3)$: Define

$S = \{v \in V \mid \exists \text{ path p from s to v in } G_f\}$

$T = V \setminus S$ (note $t \notin S$ according to (2))

residual network G_f



Cuts of Flow slides prepared by Shwetha and Pradeep

F. Fulkerson: Max-flow min-cut theorem

If f is a flow in a flow network $G = (V,E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

proof:

$(2) \Rightarrow (3)$: Define

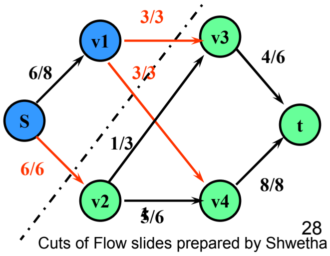
$S = \{v \in V \mid \exists \text{ path p from s to v in } G_f\}$

$T = V \setminus S$ (note $t \notin S$ according to (2))

$\Rightarrow$ for $\forall\, u \in S, v \in T$: $f(u, v) = c(u, v)$ (otherwise $(u, v) \in E_f$ and $v \in S$)

$\Rightarrow$ $|f| = f(S, T) = c(S, T)$

original network G



Cuts of Flow slides prepared by Shwetha and Pradeep

F. Fulkerson: Max-flow min-cut theorem

If f is a fow in a flow network $G = (V,E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

proof:

(3) $\Rightarrow$ (1): $|f| = f(S, T) \leq c(S, T)$

the statement of (3) : $|f| = c(S, T)$ implies that f is a maximum flow

Cuts of Flow slides prepared by Shwetha and Pradeep

- Suppose that each source s_i in a multisource, multisink problem produces exactly p_i units of flow, so that $f(s_i,V) = p_i$. Suppose that each sink t_j consumes exactly q_j units so that $f(V,t_j) = q_j$, where . Show how to convert the problem of finding a flow f that obeys these additional constraints into the problem of finding a maximum flow in a single-source, single-sink flow network.
- Given a flow network $G = (V, E)$, let $f1$ and $f2$ be functions from $V \times V$ to **R**. The flow sum $f1 + f2$ is the function from $V \times V$ to **R** defined by $(f1 + f2)(u, v) = f1(u, v) + f2(u, v)$ for all $u, v \in V$. If $f1$ and $f2$ are flows in G, which of the three flow properties must the flow $f1 + f2$ satisfy, and which might it violate?
- The edge connectivity of an undirected graph is the minimum number k of edges that muct be removed to disconnect thegraph. For example, the edge connectivity of a tree is 1, and the edge connectivity of a cyclic chain of vertices is 2. Show that how the edge connectivity of an undirected graph $G = (V,E)$ can be determined by running a maximum-flow algorithm on at most $|V|$ flow networks, each having $O(V)$ vertices and $O(E)$ edges.