Computational Geometry

TOPICS
- Preliminaries
- Point in a Polygon
- Polygon Construction
- Convex Hulls

Further Reading

Chapter 35 from Text book

Geometric Algorithms

Geometric Algorithms find applications in such areas as

- **Computer Graphics**
- **Computer Aided Design**
- **VLSI Design**
- **GIS**
- **Robotics**

During this week we will study algorithms dealing with points, lines, line segments, and polygons

In particular, the algorithms will

- **Determine whether a point is inside a Polygon**
- **Construct a Polygon**
- **Determine Convex Hulls**

Preliminaries:

A point p is represented as a pair of coordinates (x,y)
A line is represented by a pair of points
A path is a sequence of points $p_1, p_2, \ldots p_n$ and the line segments connecting them,

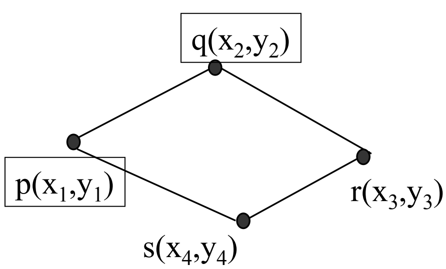
p_1-p_2, p_2-p_3, . . . , p_{k-1}-p_k.

A closed path whose last point is the same as the first is a polygon.

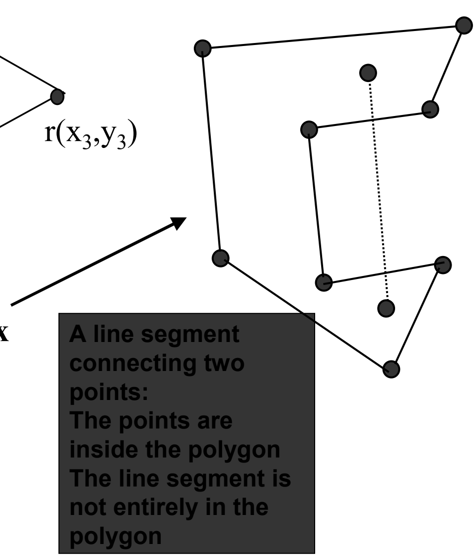
A simple polygon is one whose corresponding path does not intersect itself. It encloses a region in the plane.

A Convex Polygon is a polygon such that any line segment connecting two points inside the polygon is itself entirely in the polygon.

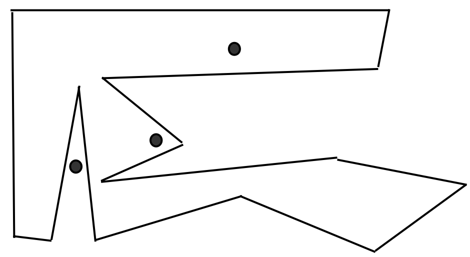
The convex hull of a set of points is defined as the smallest convex polygon enclosing all the given points.



This is not a convex polygon



Determining whether a *point* is inside a polygon



Given a simple polygon polygon P, and a point q, determine whether the point is inside or outside the polygon. (a non-convex polygon)

Procedure **Point_in_a_Polygon(P,q)**

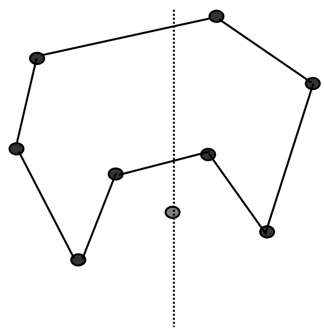
Input : P (a simple polygon with vertices p_1,p_2,p_3, and edges e_1,e_2,e_3, … e_n and q (x_0,y_0) a point.

Output: INSIDE (a Boolean variable, True if q is inside P, and false otherwise)

```
Count ← 0;
  for all edges e_i of the polygon do
    if the line x = x_0 intersects e_i then
      y_i ← y coordinate of the intersection between lines e_i and x=x_0;
      if y_i > y_0 then
        Count ← Count +1;
  if count is odd then INSIDE ← TRUE;
  else INSIDE ← FALSE
```

This does not work if the line passes through terminal points of edges

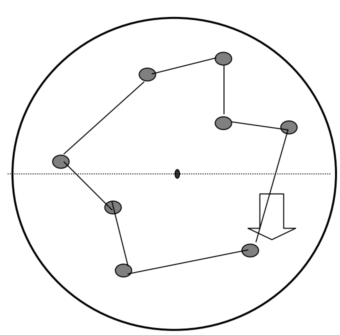
It takes constant time to perform an intersection between two line segments.
The algorithm computes n such intersections, where n is the size on the polygon.
Total running time of the algorithm, O(n).



Constructing a Simple Polygon

Given a set of points in the plane, connect them in a simple closed path.

> **Consider a large circle that contains all the points.**
> **Scan the area of C by a rotating line.**
> **Connect the points in the order they are encountered in the scan.**



Procedure Simple_Polygon

Input : $p_1, p_2, \ldots p_n$ (points in the polygon)
Output : P (a simple polygon whose vertices $p_1, p_2, \ldots p_n$ are in some order)

1. for $i \leftarrow 2$ to n
2. $\alpha_i \leftarrow$ angle between line p_1-p_i and the x-axis;
3. sort the points according to the angles
(use the corresponding priority for the point and do a heapsort)
4. P is the polygon defined by the list of points in the sorted order.

Complexity : Complexity of the sorting algorithm.

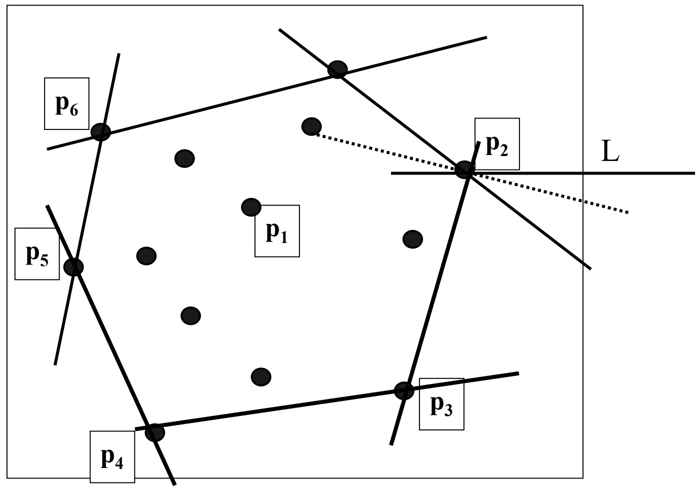
Convex Hulls

The convex hull of a set of points is defined as the smallest convex polygon enclosing all the points in the set.

The convex hull is the smallest region encompassing a set of points.
A convex hull can contain as little as three and as many as all the points as vertices.

Problem Statement : Compute the convex hull of n given points in the plane.

There are two algorithms
Gift Wrapping $O(n^2)$
Graham's Scan O(nlogn)

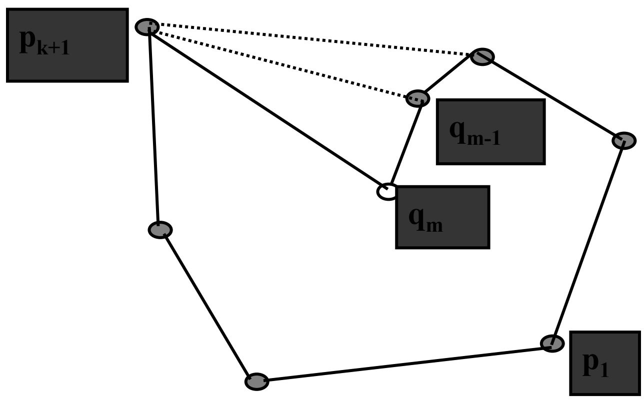


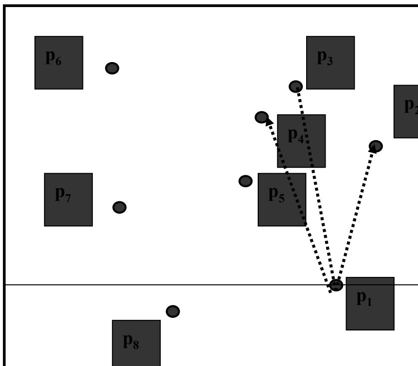
Procedure Gift_Wrapping(p_1,p_2, . . . p_n)
Input : p_1,p_2, . . . p_n (a set of points in the plane)
Output : P (the convex hull of p_1,p_2, . . . p_n)

1. P ← {0} or ε;
2. p ← a point in the set with the largest x-coordinate;
3. Add p to P;
4. L ← line containing p and parallel to the x-axis;
5. while |P| < n do
6. q ← point such that the angle between the line -p-q- and L is minimal among all points;
7. add q to P;
8. L ← line -p-q-;
9. p←q;

Graham's Scan:

Given a set of n points in the plane, ordered according to the algorithm Simple Polygon, we can find a convex path among the first k points whose corresponding convex polygon encloses the first k points.



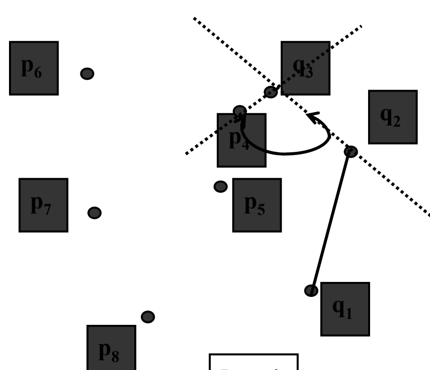


q1:p1

q2:p2

q3:p3

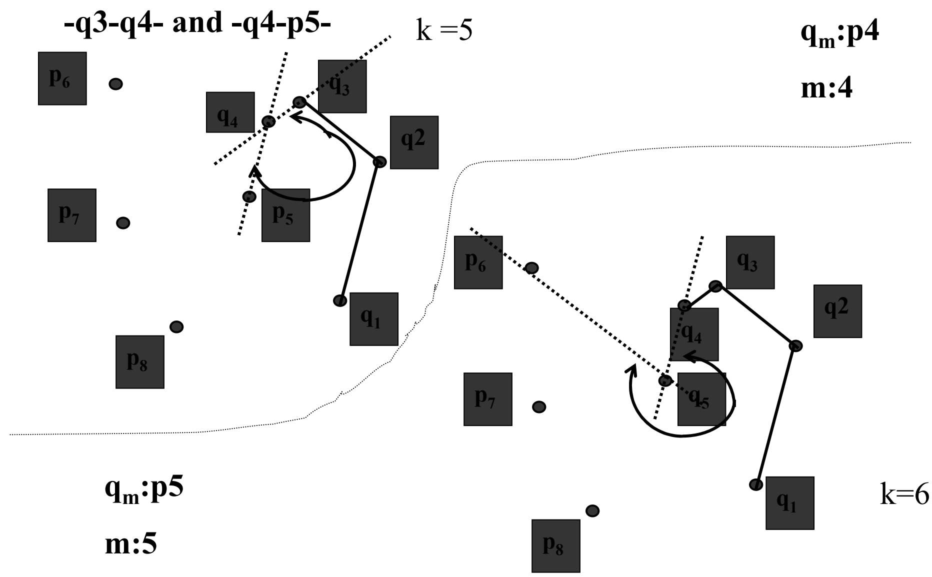
m:3

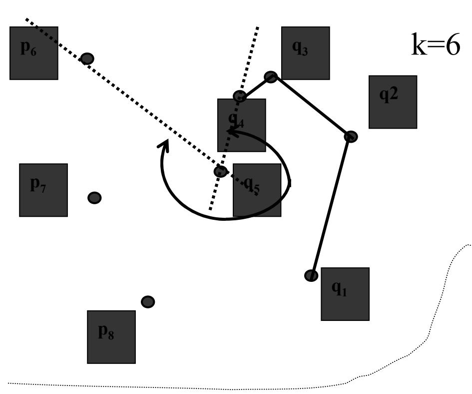


-q2-q3- and -q3-p4-

The angle is < 180

k =4





Angle between -q4-q5- and -q5-p6- is greater than 180

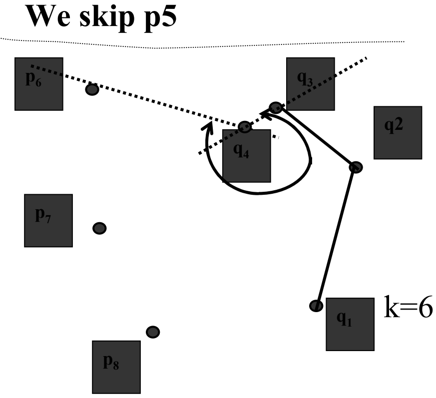
Therefore m = m-1 = 4

We skip p5

Angle between -q3-q4- and -q4-p6- is greater than 180

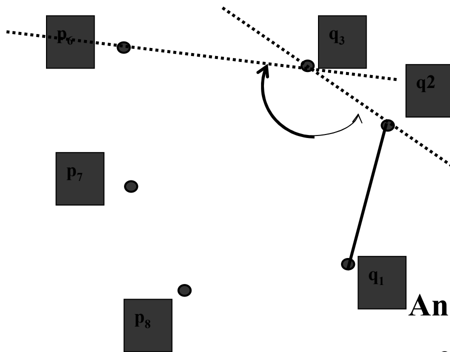
Therefore m = m-1 = 3

We skip p4



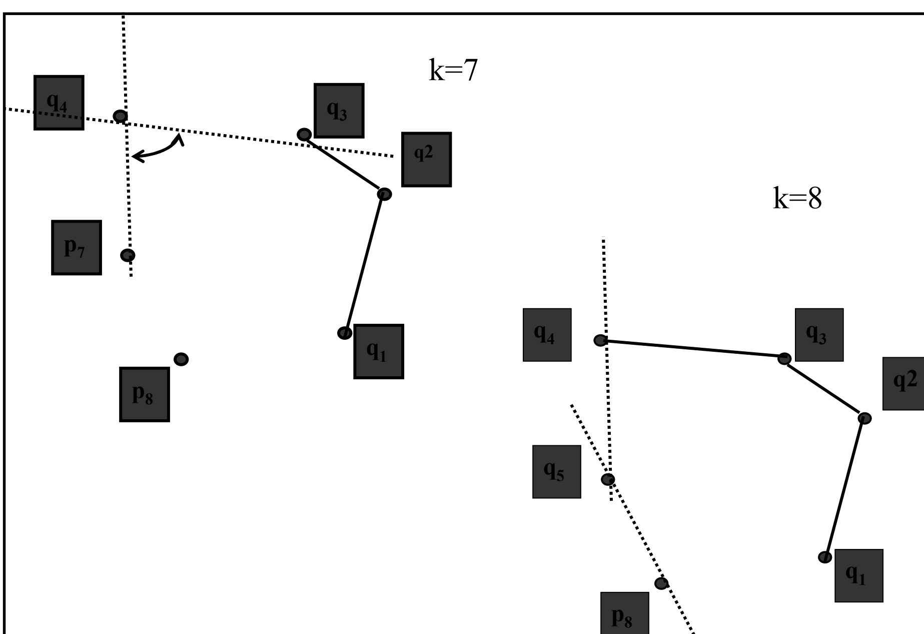
-q3-q4- and -q4-p6-

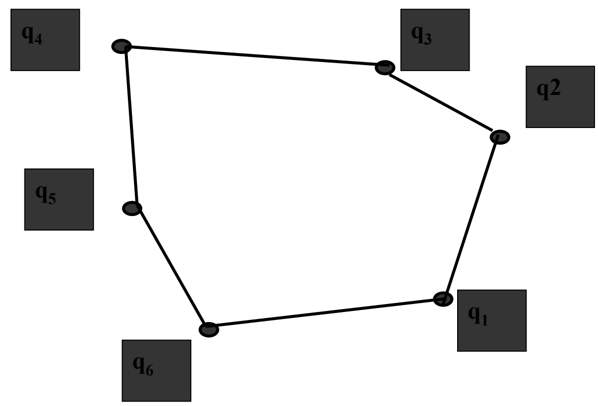
k=6



Angle between -q2-q3- and -q3-p6- is less than 180

Therefore m = m+1 = 4 and q4 = p6





Procedure Graham's Scan($p_1,p_2, \ldots p_n$)

Input : $p_1,p_2, \ldots p_n$(a set of points in the plane)

Output : $q_1,q_2, \ldots q_n$(the convex hull of $p_1,p_2, \ldots p_n$)

p1 $\leftarrow$ the point in the set with the largest x-coordinate
(and smallest y-coordinate if there are more than one point with the same x-coordinate)

Construct Simple Polygon and arrange points in order

Let order be $p_1,p_2, \ldots p_n$

$q_1 \leftarrow p_1$;

$q_2 \leftarrow p_2$;

$q_3 \leftarrow p_3$;

(initially P consists of p_1,p_2,and p_3)

$m \leftarrow 3$;

for $k \leftarrow 4$ to n do

while the angle between lines $-q_{m-1}-q_m-$ and $-q_m-p_k- \geq 180^o$ do

$m \leftarrow m-1$; [Internal to the polygon]

$m \leftarrow m+1$;

$q_m \leftarrow p_k$;